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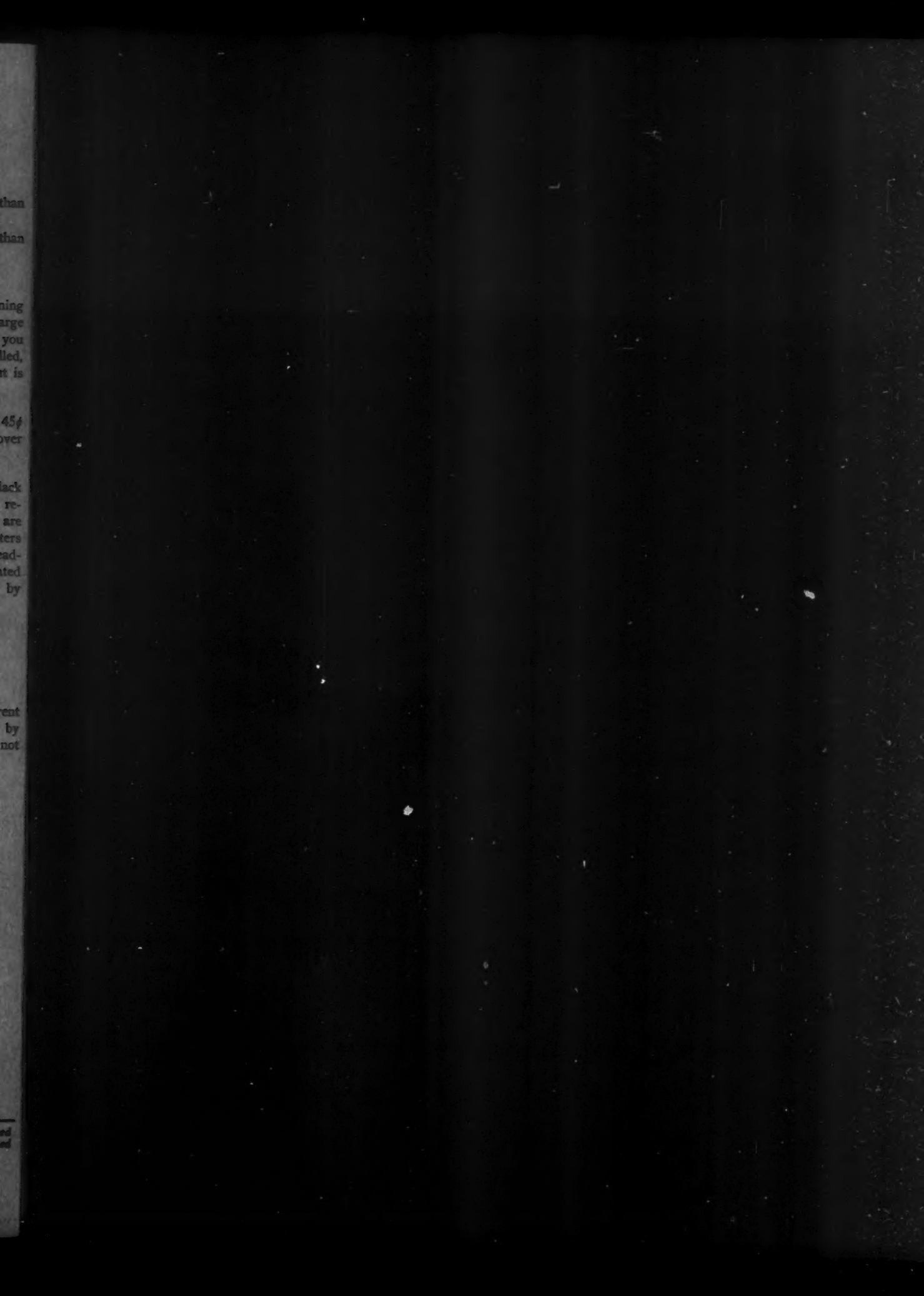


Mathematical Reviews

Vol. 5, No. 3 MARCH, 1944 Pages 57-84

HISTORY

Neubauer, F. J. An old astronomical manuscript. Publ. Astr. Soc. Pacific **55**, 145–146 (1943). [MF 8484]

"The manuscript is a copy made about 1420 A.D. of a much older text written about 1230 A.D. in Paris by Johannes de Sacrobosco." It "deals largely with the computation of the calendar, the subject of lunations," etc. "The first printed copy of our manuscript appeared in Wittenberg in 1538 A.D."

Ludendorff, H. Astronomische Inschriften in Piedras Negras und Naranjo. Untersuchungen zur Astronomie der Maya, Nr. 13. Abh. Preuss. Akad. Wiss. Math.-Nat. Kl. **1940**, no. 6, 60 pp. (1940). [MF 8678]

***Carmody, Francis J. Thabit b. Qurra, Four Astronomical Tracts in Latin.** Berkeley, Calif., 1941. 28 pp.

Edition of four short treatises in medieval Latin which are attributed to Tabit b. Qurra. The first, "De motu octavae sphaerae," deals mainly with precession; the second is a short list of explanatory remarks for the reading of the Almagest. There follows a brief introduction to spherical astronomy and a discussion of sizes and distances of the celestial bodies. *O. Neugebauer* (Providence, R. I.).

Miller, G. A. A sixth lesson in the history of mathematics Nat. Math. Mag. **17**, 341–350 (1943). [MF 8438]

Miller, G. A. A seventh lesson in the history of mathematics. Nat. Math. Mag. **18**, 67–76 (1943). [MF 9557]

Dehn, Max. Mathematics, 600 B.C.–400 B.C. Amer. Math. Monthly **50**, 357–360 (1943). [MF 8486]

Dehn, Max. Mathematics, 400 B.C.–300 B.C. Amer. Math. Monthly **50**, 411–414 (1943). [MF 9089]

Richards, John F. C. A new manuscript of arithmomachia. Scripta Math. **9**, 87–99 (1943). [MF 9027]

Levi, B., Capelli, Pedro and Cotlar, Mischa. The origins of the theory of the Wronskian algorithm within the Pythagorean doctrine. Math. Notae **3**, 74–100 (1943). (Spanish) [MF 9249]

Hofmann, Jos. E. Über ein "neues" Verfahren zur Annäherung von Quadratwurzeln und seine geschichtliche Bedeutung. Deutsche Math. **6**, 453–461 (1942). [MF 8619]

The expansion of $\sqrt{k}$ by a continued fraction is modified by selecting certain subsequences with better convergence. Archimedes' inequality $265/153 < \sqrt{3} < 1351/780$ can be obtained by this method. *O. Neugebauer.*

Boyer, C. B. An early reference to division by zero. Amer. Math. Monthly **50**, 487–491 (1943). [MF 9260]

Dassen, C. C. Investigations, with illustrations, concerning the mathematician who solved the equations of third degree and founded the theory of them. An. Soc. Ci. Argentina **134**, 134–170 (1942). (Spanish) [MF 8430]

Whittaker, E. T. Aristotle, Newton, Einstein. Philos. Mag. (7) **34**, 266–280 (1943). [MF 8412]

Vonwiller, O. U. Galileo and Newton: their times and ours. J. Proc. Roy. Soc. New South Wales **76**, 316–328 (1943). [MF 9226]

Andrade, E. N. da C. Newton and the science of his age. Proc. Roy. Soc. London. Ser. A. **181**, 227–243 (1943). [MF 8477]

Jeans, James. Newton and the science of to-day. Proc. Roy. Soc. London. Ser. A. **181**, 251–262 (1943). (One plate) [MF 8478]

Sergescu, Pierre. Sur l'identité des auteurs de quelques articles de mathématiques, publiés de 1692 à 1703 dans le Journal des Savants. C. R. Acad. Sci. Paris **214**, 971–973 (1942). [MF 9470]

[A longer article under the same title appeared in An. Acad. Romane. Mem. Sect. Stiintifice (3) **17**, no. 9, 21 pp. (1942); these Rev. **4**, 65.]

Crawford, L. Edward Waring, eighteenth century mathematician. Trans. Roy. Soc. South Africa **29**, 69–74 (1942). [MF 8784]

Hofmann, Jos. E. Über zwei geodätische Briefe von C. F. Gauss aus dem Jahr 1800. Deutsche Math. **6**, 576–585 (1942). [MF 8606]

Two recently discovered letters are here reproduced and discussed. The first letter contains a rather simple problem concerning the convergence of meridians. The second letter shows that Gauss had discovered as early as 1800 the so-called method of Soldner. Both letters are answers to questions raised by Lecoq in connection with practical surveying. *O. Neugebauer* (Providence, R. I.).

Mandelbrojt, S. Les mathématiques en France au XIX• siècle. Rev. Trimest. Canad. **29**, 253–258 (1943). [MF 8962]

Piaggio, H. T. H. The significance and development of Hamilton's quaternions. Nature **152**, 553–555 (1943). [MF 9710]

Bateman, H. The influence of tidal theory upon the development of mathematics. Nat. Math. Mag. **18**, 14–26 (1943). [MF 9358]

Planck, Max. Zur Geschichte der Auffindung des physikalischen Wirkungsquantums. Naturwissenschaften **31**, 153–159 (1943). [MF 8499]

Coolidge, J. L. Three hundred years of mathematics at Harvard. Amer. Math. Monthly **50**, 347–356 (1943). [MF 8485]

Neville, E. H. Obituary: Andrew Russell Forsyth. 1858–1942. J. London Math. Soc. **17**, 237–256 (1942). [MF 8763]

Ford, W. B. Obituary: Earle Raymond Hedrick. Amer. Math. Monthly **50**, 409–411 (1943). [MF 9088]

Sommerfeld, A. Zum Andenken an David Hilbert. Naturwissenschaften **31**, 213–214 (1943). [MF 8676]

Cartan, Élie. Notice sur M. Tullio Levi-Civita. C. R. Acad. Sci. Paris **215**, 233–235 (1942). [MF 9490]

Hodge, W. V. D. Obituary: Tullio Levi-Civita. J. London Math. Soc. **18**, 107–114 (1943). [MF 9705]

Ruse, H. S. Obituary: Tullio Levi-Civita. Edinburgh Math. Notes no. **33**, 19–24 (1943). [MF 9337]

Birkhoff, George D. Obituary: William Fogg Osgood. Scientific Monthly **57**, 466–469 (1943). [MF 9327]

Coolidge, Julian L., Birkhoff, George D. and Kemble, Edwin C. Obituary: William Fogg Osgood. Science **98**, 399–400 (1943). [MF 9328]

Hadamard, J. Obituary: Émile Picard. J. London Math. Soc. **18**, 114–128 (1943). [MF 9706]

Onicescu, O. Obituary: D. Pompeiu. Mathematica, Timişoara **19**, 12–15 (1943). [MF 9427]

Maier, W. Obituary: Karl Reinhardt. Jber. Deutsch. Math. Verein. **52**, 75–83 (1942). [MF 8536]

Rosenblatt, Alfred. Obituary: Vito Volterra. Revista Ci., Lima **44**, 423–442 (1942). (Spanish) [MF 8748]

Hardy, G. H. Obituary: William Henry Young. J. London Math. Soc. **17**, 218–237 (1942). [MF 8762]

THEORY OF GROUPS

Miller, G. A. Elements of the generality of the group concept. Science **98**, 362–363 (1943). [MF 9278]

Miller, G. A. Groups of transformations of the noninvariant subgroups. Proc. Nat. Acad. Sci. U. S. A. **29**, 240–242 (1943). [MF 8954]

Quoting the result that a regular group of prime order is the group of transformations of all the noninvariant subgroups of some group (property *), the author goes on to prove that no regular group of composite order has this property *. Further no nonregular imprimitive group has this property *. In fact, a necessary and sufficient condition that a nonregular primitive group have the property * is that its order be the product of two distinct prime numbers such that the larger diminished by one is divisible by the smaller. *G. de B. Robinson* (Ottawa, Ont.).

Levi, F. W. Contributions to the theory of ordered groups. Proc. Indian Acad. Sci., Sect. A. **17**, 199–201 (1943). [MF 9265]

Every ordered group satisfies the condition (a) "every element not equal to 1 has infinite order." If a group satisfies (a) and (b) "the commutator group lies in the center," then it can be ordered. The author shows that (a) is not a sufficient and (b) not a necessary condition. He also proves another necessary condition which is more restrictive than (a). *H. B. Mann* (Barrytown, N. Y.).

Neumann, B. H. Adjunction of elements to groups. J. London Math. Soc. **18**, 4–11 (1943). [MF 9204]

The problem discussed in this paper is that of finding a group which contains a preassigned group S as subgroup and which contains elements meeting certain requirements (equations with "coefficients" in S). The problem differs from the customary extension problem in that the extended group need not be a normal subgroup of the extension and that the conditions are not formulated in a form involving the quotient system modulo the extended group.

R. Baer (Urbana, Ill.).

Neumann, B. H. On the number of generators of a free product. J. London Math. Soc. **18**, 12–20 (1943). [MF 9205]

Denote by $m(G)$ the minimum number of elements needed for generating the group G. F. W. Levi [J. Indian Math. Soc. (N.S.) **5**, 149–155 (1941); these Rev. **4**, 2] has conjectured that $m(G)=\sum_\nu m(G_\nu)$ if G is the free product of the groups G_ν and verified this conjecture in case $m(G)\leqq 2$. The note under review is devoted to a proof of the general case of this important theorem. It is readily seen that it suffices to consider the special case of a free product G of two groups A and B such that both $m(A)$ and $m(B)$ are finite. It is shown that a system of $m(G)$ generators of G, meeting certain minimum and maximum requirements, contains both a system of generators of A and a system of generators of B; and from this fact one readily deduces $m(G)=m(A)+m(B)$. Another proof of this theorem has been given by I. Gruschko [Rec. Math. [Mat. Sbornik] N.S. **8**(50), 169–182 (1940); these Rev. **2**, 215].

R. Baer (Urbana, Ill.).

Venkatarayudu, T. Normal co-ordinates of symmetric point groups. Proc. Indian Acad. Sci., Sect. A. **17**, 50–54 (1943). [MF 8519]

Characters are determined for the class of symmetry groups G generated by two elements P and Q such that $P^n=E$, $Q^2=E$ and $PQ=QP^{-1}$ and also for groups of the form $G\times H$ (the direct product of G and another group of order 2). It is then indicated how these results may be applied to the problem of determining normal coordinates for the modes of oscillation of a molecule in case degeneracy exists. *A. H. Taub* (Princeton, N. J.).

Venkatarayudu, T. The characters of the classes $(n-k, k)$ of the symmetric group of degree n. J. Indian Math. Soc. (N.S.) **7**, 42–45 (1943). [MF 9380]

Pursuing the methods of D. E. Littlewood, the author gives formulae for the characters in question.

G. de B. Robinson (Ottawa, Ont.).

Baer, Reinhold. A theory of crossed characters. Trans. Amer. Math. Soc. **54**, 103–170 (1943). [MF 8713]

The author considers a group G of finite order and a cyclic group E of order m. He assumes that a fixed homomorphism C of G into the group of automorphisms of E is given. Thus, to every element g of G there corresponds a mapping of E on itself; the image of the element c of E is denoted by c^g. A C-character (or crossed character) is a function $f(g)$ of an arbitrary element of G whose values lie in E and which satisfies the equation $f(uv)=f(u)^v f(v)$ for any two u, v in G. In the special case where C maps every element of G on the identical automorphism of E, the C-character f may be considered as an ordinary linear character of G. We may, therefore, ask which of the properties of ordinary linear characters of groups, known from the theory of Abelian groups of finite order, will hold for the C-characters of G. It is a generalization of this theory, not a generalization of the general theory of group characters, with which the author is concerned. Incidentally, for any C-character f we have $f(u)=1$ for the elements u of the second commutator subgroup G'' of G, as is easily seen. Thus the investigations of the author belong to the study of metabelian groups. Because of the great length of the paper we must restrict ourselves to an indication of the scope of the contents without being able to go into any detail. A group G is said to be C-complete if 1 is the only element in G mapped upon 1 by every C-character of G. A number of different sets of necessary and sufficient conditions for C-completeness are developed. The C-characters of G form an Abelian group L. The order of this C-character group L is determined. For C-complete groups G, the necessary and sufficient condition is established that G and the C-character group L both have the same order. A subgroup B of L is said to be complete if 1 is the only element mapped upon 1 by every C-character of B. The group B is said to be strictly complete if B is complete and contains all C-characters $f(g)=c^{1-g}$ (c in E, fixed). Conditions are given which guarantee that the full C-character group L is the only complete group B (is the only strictly complete group B). Finally, the groups G are determined for which a duality between G and L can be established; also, all groups G are obtained which are isomorphic to L. The author mentions that he will apply his results to radical extensions of fields in a later paper. *R. Brauer.*

Smith, P. A. Foundations of the theory of Lie groups with real parameters. Ann. of Math. (2) **44**, 481–513 (1943). [MF 8876]

This paper is devoted to proving a result announced by E. R. van Kampen in 1936: every locally Euclidean r-dimensional continuous local group G_r whose operation ab (1) is left-differentiable and (2) satisfies a right Lipschitz condition (r.L.c.) is locally isomorphic to a local r-parameter group with analytic canonical coordinates. The r.L.c. serves to make the inverse of the mapping into canonical coordinates single valued and reads: a coordinate system for G_r satisfies a r.L.c. if there is a nucleus V and a $C>0$ such that $|ab-ac|<C|b-c|$ for a,b,c in V. The absolute values are the Euclidean ones in the given coordinate system. Thus the result of Garrett Birkhoff [Trans. Amer. Math. Soc. **43**, 61–101 (1938)] which requires right and left-differentiability is strengthened in the locally Euclidean finite dimensional case. In addition it is here proved that every canonical coordinate system is analytic; linearly independent vector fields of class 1 determine groups G_r and vice versa; a flat subspace H of G_r whose Lie algebra $L(H)$ is a subalgebra of $L(G_r)$ is a local subgroup of G_r, and if one of H, $L(H)$ is normal so is the other.

Under condition (1) alone it is shown that $ab=a+b$ in first approximation, and that, for a near enough the identity I, a has a unique principal square root nearer I. An application of the topological mapping theorem for spheres facilitates proof here. Then by dyadic approximation a^t, $0\leqq t\leqq 1$, is defined and shown to yield a one-parameter subgroup with a tangent vector $\Lambda(a)$ at $t=0$. The correspondence k: $a\to\Lambda(a)$ is called canonical if its inverse is single-valued; if k is canonical G_r is locally isomorphic to a local group with canonical coordinates $\Lambda(a)$ (result A). In canonical coordinates, $a^{-1}ba$ is linear in b and analytic in a (result B). If $\phi_1, \cdots, \phi_r$ are vector fields of class $k\geqq 1$, nonsingular at 0, over a neighborhood of 0, and if $\psi_1, \cdots, \psi_r$ are linearly independent vector fields, over a Euclidean S_n, such that $dx^i/dt=c^j\psi_j$ define a single valued $f(x,t)$ over an open subset of S_n for each x and set of numbers c, then

$$\partial(yx)/\partial y^j=\sum_\lambda \psi_\lambda{}^i(yx)\phi_j{}^\lambda(y)$$

define a yx which turns E_r into a G_r whose coordinates are analytic if the ϕ_i are analytic (result C). If (2) is satisfied, k is canonical and vectors

$$\psi_j=(\partial(ab)/\partial a^j)_0$$

define a single valued $f(x,t)$ [van Kampen], so that canonical coordinates exist (A) and, by means of (B), analytic ϕ_i can be found such that (C) gives the principal theorem.

W. W. Flexner (Ithaca, N. Y.).

Malcev, A. Sur les groupes topologiques locaux et complets. C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 606–608 (1941). [MF 9611]

The problem whether to a given local topological group (that is, a topological group germ) there exists a topological group locally isomorphic to it is answered by the author in the negative. The author considers the symmetric groupoids (in his own sense) and lays emphasis on the general associative law, for example, the associative law concerning products of n elements. A groupoid is said to be of rank $n\geqq 1$ with respect to a subgroupoid T if (1) the product of n elements in T is determined in S no matter how the parentheses are inserted, (2) the general associative law does not hold in T. By proving that there exists a groupoid of an arbitrary rank, it is easy to construct an example of a local topological group for which the general associative law holds for no neighborhood of the identity, and which is therefore not locally isomorphic to a topological group. It is also proved that the validity of the general associative law in a groupoid is a sufficient condition for its local isomorphism with a topological group. *S. Chern.*

von Kerékjártó, B. Über die dreigliedrigen integrierbaren Gruppen. Math. Ann. **118**, 365–378 (1942). [MF 8935]

It is shown that every solvable three parameter topological group can be realized as a group of affine transformations in Euclidean 3-space. The realizing groups, seven in number, of which two involve an arbitrary constant, are given explicitly. The author appears to be unfamiliar with Chevalley's theorem that every n-parameter solvable topological group is a Lie group [Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 291–292; these Rev. **3**, 135]. This theorem makes possible the

use of infinitestimal transformations and the result in question, together with certain of the theorems leading up to it, would seem to follow immediately from Lie's own description of the 3-dimensional solvable infinitesimal algebras [Lie-Engel, Theorie der Transformationsgruppen, vol. 3, Teubner, Leipzig, 1893]. *P. A. Smith.*

de Kerékjártó, B. Sur la caractère topologique du groupe homographique de la sphère. J. Math. Pures Appl. (9) **21**, 67–100 (1942). [MF 9297]

Theorem 1. Every connected two-parameter topological group is equivalent to a Lie group. Theorem 2. A doubly transitive topological transformation group on the plane is equivalent to the group of complex transformations $z'=az+b$. Theorem 3. A triply transitive topological transformation group on the sphere is equivalent to the group of complex transformations

$$z'=(az+b)/(cz+d).$$

The proof of theorem 1 leans rather heavily on Brouwer's translation theorem and its corollaries. *P. A. Smith.*

Montgomery, Deane and Samelson, Hans. Transformation groups of spheres. Ann. of Math. (2) **44**, 454–470 (1943). [MF 8874]

Two theorems are first proved together by induction on n. Theorem I. If the compact connected Lie group G acts transitively and effectively on an n-sphere S^n, then (a) if n is even, G is simple; (b) if n is odd, G is either simple or can be written $(G_1\times G_2)/N$, where G_i are simple, G_2 is either the rotation group R_1 of S^1 or the universal covering group U_1 of R_2, N is a finite normal subgroup and the subgroup of G corresponding to G_1 is transitive on S^n. Theorem I'. If $G=(G_1\times G_2)/N$, where G_i are compact connected Lie groups, N is as above and G is transitive on S^n, then the subgroup of G corresponding either to G_1 or G_2 is transitive on S^n. The possibilities of I(b) occur: S^{2n+1}, $n>0$, has a group $G_1\times R_1/N$; S^{4n-1}, $n>0$, has a group $G_1\times U_2/N$ and S^{4n+1} has not.

Using the Killing-Cartan classification of simple groups it is then shown that the only simple groups transitive and effective on S^n are those already known. (1) If G is a compact classical group (that is, a locally connected group locally isomorphic either to R_n, or to the unitary unimodular group A_n on $n+1$ complex variables, or to the symplectic group C_n on $2n$ complex variables) and if G is transitive on S^{2m}, then G is locally isomorphic (l.i.) to R_{2m}. (2) If G is a is a compact connected simple Lie group transitive on S^n, $n>114$ and even, then G is l.i. to R_n. (3) The only classical compact Lie groups which act transitively on S^n, $n\equiv 1(4)$, are l.i. to R_n and $A_{(n-1)/2}$. (4) If $n\equiv 3(4)$, it is true for all but a finite number of n that the only simple groups transitive on S^n are R_n, $A_{(n-1)/2}$, $C_{(n+1)/4}$. Two theorems on the structure of R_n conclude the paper.

In the proofs use is made principally of theorems on the rank of Lie groups [Hopf, Comment. Math. Helv. **13**, 119–143 (1940); these Rev. **3**, 35], on their fibre decomposition [Gysin, Comment. Math. Helv. **14**, 61–112 (1942); these Rev. **3**, 317] and on their homology rings, together with results due to Montgomery and Zippin on transformation groups and due to Samelson on group-manifolds. *W. W. Flexner* (Ithaca, N. Y.).

James, R. C. Topological groups as subgroups of linear topological spaces. Duke Math. J. **10**, 441–453 (1943). [MF 8965]

This paper studies the concepts of boundedness, convexity and normability as applied to a topological Abelian group, and in terms of these concepts it derives conditions under which such a group may be imbedded in a linear topological space. The following are the principal theorems proved. A convex topological Abelian group which can be generated by an arbitrary neighborhood of the identity can be represented as a subgroup of a linear topological space. A normed topological Abelian group can be represented as a subgroup of a Banach space. A necessary and sufficient condition for the normability of a topological Abelian group which can be generated by an arbitrary neighborhood of the identity is the existence of a bounded convex neighborhood of the identity. *D. Montgomery* (Princeton, N. J.).

Richardson, A. R. Groupoids and their automorphisms. Proc. London Math. Soc. (2) **48**, 83–111 (1943). [MF 9113]

A groupoid $\mathfrak{G}$ is a system in which a binary operation called multiplication is defined. In general, no laws for multiplication are presupposed. The group of automorphisms of $\mathfrak{G}$ is denoted by $G(\mathfrak{G})$. In the first part of the paper $\mathfrak{G}$ is simply a set and $G(\mathfrak{G})$ a group of transformations. The system of equivalence relations R in $\mathfrak{G}$ is discussed in relation to their transformation properties under $G(\mathfrak{G})$. For a given R the transformations α in $G(\mathfrak{G})$ for which $a\equiv a^\alpha$ (mod R) form a subgroup, the group of inertia of R. Special cases are the groups G_A leaving a set A elementwise fixed or the group of decomposition $G_3(A)$ of A transforming A into itself. One finds that G_A is a normal subgroup of $G_3(A)$. Equivalence relations giving rise to the same inertial groups are called isotopic. In the set of isotopic equivalence relations there exists a minimal one. The groups of inertia for various equivalence relations and their interrelations are studied in some detail.

In the second part of the paper these concepts are applied particularly to coset expansions in groupoids. The existence of such expansions requires the existence of certain weak associative laws. Conditions are determined under which the coset expansions as equivalence relations are structure isomorphic to the structure of subgroupoids. The Galois theory for subgroups of the group of automorphisms and the corresponding subgroupoids is discussed briefly. Finally factor sets and special types of automorphisms are considered. Certain automorphisms existing only in nonassociative systems should be noted. *O. Ore.*

ANALYSIS

Calculus

***Cell, John W. Engineering Problems Illustrating Mathematics.** McGraw-Hill Book Company, Inc., New York, 1943. xi+172 pp. $1.75.

***Calcagno, Horacio E. Essay on the Algebra of Annuli.** Coni, Buenos Aires, 1942. 28 pp. (Spanish)

This paper presents an extended detailed development of a previous paper [Revista Union Mat. Argentina **7**, 12–17 (1940); these Rev. **2**, 254] in which the author imposed

upon the field of complex numbers an Archimedean order by defining $x<y$ whenever (1) $|x|<|y|$, (2) $\arg x<\arg y$ in case $|x|=|y|$. The set of all points z between x and y ($x\leqq z\leqq y$) has as locus an annular region. The author develops an algebra whose elements are numbers $(A; l_x, l_y)$ of three components [see quoted review].

L. M. Blumenthal (Columbia, Mo.).

Walsh, C. E. A proof of the "theorem of means." Edinburgh Math. Notes no. 33, 17–18 (1943). [MF 9336]

An elementary proof by induction of the familiar theorem: if $a_1, a_2, \cdots, a_n$ are positive and not all equal, then $\sum_1^n a_r^n > n\prod_1^n a_r$.

O. Szász (Cincinnati, Ohio).

Levi, B. Approximations to $n!$ for large values of n. Applications. Math. Notae 3, 148–154 (1943). (Spanish) [MF 9563]

Stirling's formula is proved in a simple way by observing that $n^{n+1}e^{-n}/n!$ decreases, while $n^{n+1}e^{-n+1/12n}/n!$ increases. Wallis' formula is used in the usual way. The known formula

$$\lim e^{-n}\sum_{k=0}^{n} n^k/k! = \tfrac{1}{2}$$

is also proved anew.

W. Feller (Providence, R. I.).

Bödewadt, U. T. Die Kettenregel für höhere Ableitungen. Math. Z. 48, 735–746 (1943). [MF 9011]

Several generalizations of the formula for the derivative of a composite function are given. Included are cases typified by $d^n y/dt^n$, where

$$y=f_1(f_2(f_3(t))),\ y=g(f_1(t), f_2(t), f_3(t)),$$

as well as applications to derivatives of inverse functions.

P. Franklin (Cambridge, Mass.).

Levi, Howard. Exact nth derivatives. Bull. Amer. Math. Soc. 49, 631–636 (1943). [MF 8891]

Let y_i denote the ith derivative of $y=y(x)$ and let $\theta(x, y_0, y_1, \cdots)$ be a given function of x and of a finite number of the y_i. The problem is to determine the conditions under which the function θ is the nth derivative with respect to x of a function $\psi(x, y_0, y_1, \cdots)$ of similar nature. Let E_i be the operation

$$\sum_{i=0}^{\infty} \binom{i+t-1}{i} D^i \partial/\partial y_i,$$

where D denotes the total differentiation with respect to x:

$$D=\partial/\partial x+\sum_{k\geq 0} y_{k+1}\partial/\partial y_k;$$

then $E_t\theta=0$, $1\leqq t\leqq n$, is a necessary condition for the existence of such a function ψ. If this condition is satisfied then θ differs from the nth derivative of a function by an expression of the form $\pi_1 y_1+\cdots+\pi_{n-1}y_{n-1}$, where π_ν is a polynomial in x and the degree of π_ν is less than ν.

G. Szegö (Stanford University, Calif.).

Grüss, Gerhard. Über das θ im verallgemeinerten Mittelwertsatz der Integralrechnung. Math. Z. 48, 712–714 (1943). [MF 9009]

For $f(x)$ continuous and $p(x)$ nonnegative and integrable in $a\leqq x\leqq b$, a new proof is given that

$$\int_a^b f(x)p(x)dx = f(\xi)\int_a^b p(x)dx$$

for at least one value of ξ within the interval (a, b).

F. G. Dressel (Durham, N. C.).

Roca, Marcelo Alonso. Extensions of Green's theorem and an application. Revista Soc. Cubana Ci. Fis. Mat. 1, 63–67 (1942). (Spanish) [MF 8311]

This note is concerned with some simple results on Green's theorem in the second form when the functions involved are vectors or tensors of higher order.

W. T. Reid.

Roca, Marcelo Alonso. A new generalization of Green's theorem. Revista Soc. Cubana Ci. Fís. Mat. 1, 92–93 (1942). (Spanish) [MF 8941]

This note is concerned with the improvement of the results reviewed above.

W. T. Reid (Chicago, Ill.).

Weber, Hans R. Berechnung zweier in der Plattentheorie auftretenden bestimmten Integrale. Ing.-Arch. 13, 377–380 (1943). [MF 9075]

Evaluation, by contour integration, of

$$\int_0^{2\pi} \frac{\cos n\psi d\psi}{(\cosh\beta-\cos\psi)^2}$$

and

$$\int_0^{2\pi} \frac{\cosh\alpha-\cos(\psi-\eta)}{(\cosh\beta-\cos\psi)^2} \ln\frac{\cosh\iota-\cos(\psi-\eta)}{\cosh\alpha-\cos(\psi-\eta)}d\psi.$$

E. Reissner (Cambridge, Mass.).

Binaud, René. Sur une généralisation du théorème de Guldin. C. R. Acad. Sci. Paris 212, 41–43 (1941). [MF 9181]

Dolaptschiew, Bl. Eine einfache geometrisch-mechanische Analogie. Z. Angew. Math. Mech. 22, 164–167 (1942). [MF 8899]

Algebraic and vectorial solution of the problem of placing n mass-points, with masses in general unequal, on a circle so that their center of gravity is at the center of the circle.

P. Franklin (Cambridge, Mass.).

Babini, José. On conics of maximum contact. Revista Union Mat. Argentina 9, 96–106 (1943). (Spanish) [MF 9441]

Scott, Winston M. Some special roulettes. Bol. Mat. 16, 102–108 (1943). [MF 9131]

Harwood, Mary K. B. and Manley, R. G. On the determination of the phase angle between two sinusoidal variations of equal frequency by means of a Lissajou figure. Math. Gaz. 27, 155–158 (1943). [MF 9526]

Theory of Sets, Theory of Functions of Real Variables

***Piccard, Sophie. Sur des ensembles parfaits.** Mém. Univ. Neuchâtel, vol. 16. Secrétariat de l'Université, Neuchâtel, 1942. 172 pp.

The perfect sets studied in this memoir may be regarded as originating, via generalization, from the Cantor perfect set, as representable in ternary notation. Let n, k be integers, $n\geqq 3$, $1\leqq k<n-1$, and $0=a_0, a_1, \cdots, a_k$ $(0<a_1<a_2\cdots<a_k)$ elements of the sequence $0, 1, \cdots, n-1$. Let $\mathfrak{A}=\{a_0, a_1, \cdots, a_k\}_n$ be the set of real numbers not less than zero which can be written as n-ary developments (that is, in the system of notation of base n) with the use

of no other digits than $a_0, a_1, \cdots, a_k$. Let $A=[a_0, a_1, \cdots, a_k]_n$ be the subset of $\mathfrak{A}$ consisting of the points of it in the closed interval $(0, 1)$; $\mathfrak{A}$ and A are nondense perfect sets of measure zero. Let $\mathfrak{F}$, F be the totality of the sets $\mathfrak{A}$, A. It is $\mathfrak{F}$ and F which form the subject of the present study. Scattered articles and notes by Steinhaus, Denjoy, Sierpiński, Zarankiewicz and others closely relate to the present work as does notably the author's tract "Sur les ensembles de distances . . ." [Mém. Univ. Neuchâtel, vol. 13, Neuchâtel, 1939; these Rev. 2, 129]. However, the ideas and results are, on the whole, independent and new.

To indicate the specific content, we list sample results from the various chapters. Chapter I [pp. 13–30] (structure and some general properties of $\mathfrak{A}$, A): necessary and sufficient conditions that A be of "first species" (defined by classifying the complementary intervals into ranks related to n-ary developments), that A possess a center of symmetry. The representation $A=[a_0, a_1, \cdots, a_k]_n$ is unique. If $\mathfrak{A}=\{a_0, a_1, \cdots, a_k\}_n$, $\mathfrak{A}'=\{a_0', a_1', \cdots, a_k'\}_n$ are two sets of $\mathfrak{F}$ such that the sets $\{a_0, a_1, \cdots, a_k\}$ and $\{a_0', a_1', \cdots, a_k'\}$ are symmetric to each other, then $D(\mathfrak{A})=D(\mathfrak{A}')$, $D(M)$ denoting, for a linear set M, the "set of distances" of M, that is, the set of all the numbers $|x-y|$, $x\in M$, $y\in M$. Chapter II [pp. 31–58] (L-measure of $D(\mathfrak{A})$ and $D(A)$). If $a_k<n-1$, a necessary and sufficient condition that mes $D(A)\neq 0$ is that $D(\mathfrak{A})$ contain a positive number. Conditions ensuring the simultaneous validity of mes $D(\mathfrak{A})>0$ and $D(\mathfrak{A})\neq<0, \infty)$, where $<0, \infty)$ denotes the set of nonnegative real numbers. Various existence theorems. Necessary and sufficient conditions that mes $D(\mathfrak{A})=0$. Relationship of mes $D(\mathfrak{A})$ and mes $D(A)$. Chapter III [pp. 59–85] (some properties of $\mathfrak{A}$, A, $D(\mathfrak{A})$, $D(A)$). If mes $D(\mathfrak{A})=0$ there exist two sets of $\mathfrak{F}$ respectively contained in, containing $D(\mathfrak{A})$. Every set A of F contains a nondenumerable subset E such that $D(E)\subset A$. Necessary and sufficient conditions that a set of the family $\mathfrak{F}_n$, F_n be the set of distances of a set of $\mathfrak{F}_n$, F_n; here $\mathfrak{F}_n$, F_n, for $n\geqq 3$ a fixed number, is the subset of $\mathfrak{F}$, F consisting of the sets of this family of form $\{a_0, a_1, \cdots, a_k\}_n$, $[a_0, a_1, \cdots, a_k]_n$, respectively.

Chapter IV [pp. 86–96] (measure of $s(\mathfrak{A}, \mathfrak{A}')$ and $s(\mathfrak{A})$; $s(\mathfrak{A}, \mathfrak{A}')$ is the set of real numbers $a+a'$, $a\in\mathfrak{A}$, $a'\in\mathfrak{A}'$; $s(\mathfrak{A})=s(\mathfrak{A}, \mathfrak{A})$). If $s(\mathfrak{A}, \mathfrak{A}')\neq<0, \infty)$, mes $Cs(\mathfrak{A}, \mathfrak{A}')>0$, C denoting complement. Necessary and sufficient conditions that mes $s(\mathfrak{A}, \mathfrak{A}')=0$. Examination of different types of $s(\mathfrak{A}, \mathfrak{A}')$. There exists an infinite number of sets $\mathfrak{A}$, A such that $s(\mathfrak{A})$, $s(A)$ belongs to $\mathfrak{F}$, F, respectively. Chapter V [pp. 99–172] (intersection of A and one of its translations $A[t]$; t is a given real nonnegative number, $A[t]$ the set of numbers $a+t$, $a\in A$). Here A_1, A_2, A_3 denotes the set of numbers t of $D(A)$ such that $AA[t]$ is respectively finite, denumerable, nondenumerable. Necessary and sufficient conditions that A_1 be finite. If t of $D(A)$ has a finite n-ary development, $AA[t]$ is either finite or nondenumerable. Necessary and sufficient conditions that $AA[t]$ be denumerable. Various sufficient conditions for the nondenumerability of $AA[t]$. If $n>2$, $1\leqq k<n-1$, and $A=[0, 1, 2, \cdots, k]_n$, then $A_2=0$. Theorems on the cardinals of A_1, A_2, A_3 (A_3 always nondenumerable). Many numerical illustrations.

Supplementary note [pp. 173–194] (various results for $s(a)$; here A is any linear set). There exists a linear set A such that mes $D(A)=0$, $s(A)=<0, \infty)$; $D(A)=<0, \infty)$, mes $s(A)=0$. There exist two linear bounded perfect sets A, B such that mes $s(A)=$mes $s(B)=0$, mes $s(A, B)>0$; mes $s(A)=$mes $s(B)>0$, mes $s(A, B)=0$. There are 2^c sets $s(A)$. Necessary and sufficient conditions that a finite set be an $s(A)$. If A is measurable and of positive measure, $s(A)$ contains an interval. *H. Blumberg* (Columbus, Ohio).

Piccard, Sophie. Résolution d'un problème de M. Zarankiewicz pour une famille d'ensembles parfaits. Mathematica, Timişoara 19, 26–53 (1943). [MF 9419]

Using the notation of the memoir reviewed above, the object of this article is to establish a necessary and sufficient condition that a set A of F be such that it have a denumerable number of points in common with its translation $A[t]$, $0<t<1$. The principal results are comprised in chapter 5 of the above-mentioned memoir, but the present paper gives complete details (except for the proof of 5 lemmas). The lengthy argument depends on an elaborate classification into cases. The solution of the problem of Zarankiewicz, which relates to the Cantor perfect set, is thus extended to a certain denumerable class of perfect sets. *H. Blumberg* (Columbus, Ohio).

Sard, Arthur. The equivalence of n-measure and Lebesgue measure in E_n. Bull. Amer. Math. Soc. 49, 758–759 (1943). [MF 9319]

Let A be a set in Euclidean n-space E_n, $\{A_i\}$ be a countable covering of A by unconditioned sets, $\sigma=\sum c\delta(A_i)^n$, where c is the n-volume of a sphere of unit diameter in E_n and $\delta(A_i)$ is the diameter of A_i, and $L(A, \alpha)$, for $\alpha>0$, be the greatest lower bound of all sums σ corresponding to coverings for which $\delta(A_i)<\alpha$ for all i. Then the n-measure of A is defined as $\lim_{\alpha\to 0} L(A, \alpha)$. It is shown that n-measure and outer Lebesgue measure are equal. *H. Blumberg.*

Favard, M. J. Sur la mesure dans les espaces compacts, semi-compacts ou séparables. J. Math. Pures Appl. (9) 21, 277–288 (1942). [MF 9305]

The paper describes a method for defining a measure in a compact, semi-compact or separable space. The basic construction is formulated for a compact space of finite dimension, but the resulting measure need have no connection with the dimension and may be confined to a point. *J. C. Oxtoby* (Bryn Mawr, Pa.).

Youngs, J. W. T. The additivity of the Lebesgue area. Bull. Amer. Math. Soc. 49, 779–784 (1943). [MF 9322]

The writer gives a simple proof of the following theorem. Let S be a surface represented on the unit square Q. Let Q be subdivided into two rectangles R_1 and R_2 by a segment $u=c$, $0<c<1$, say, and let S_1 and S_2 be the corresponding subsurfaces of S. Suppose that the vector function representing S is constant along the segment $u=c$. Then $L(S)=L(S_1)+L(S_2)$, where $L(S)$, etc., denote the Lebesgue areas of the respective surfaces. As the writer points out, this result is a special case of a more general result proved by the reviewer [Amer. J. Math. 58, 312–322 (1936)], an error in the reviewer's proof having been rectified by Radó and Reichelderfer [Amer. J. Math. 61, 645–650 (1939); these Rev. 1, 8]. The essential point of the proof is the lemma which shows that any polyhedron represented on Q in a quasi-linear way so that the representing vector $x(u, v)$ satisfies $|x(1, v)-a|<\epsilon/2$ may be replaced by another polyhedron $x=\bar{x}(u, v)$ ($\bar{x}$ being quasi-linear) with the same area so that $|x(u, v)-\bar{x}(u, v)|<\epsilon$ on Q and $x(1, v)=a$. *C. B. Morrey, Jr.* (Aberdeen, Md.).

Yagi, Fumio. A convergence theorem for Lebesgue-Stieltjes integrals. Bull. Amer. Math. Soc. 49, 760–767 (1943). [MF 9320]

Let V be the class of right-continuous functions $p(x)$ of bounded variation on the infinite interval $(-\infty, \infty)$. Denote by $V(p)$ the total variation of p over $(-\infty, \infty)$. The author proves the following theorem. Let $g(x)$ be a bounded Borel measurable function defined everywhere on $(-\infty, \infty)$. Let $p_n(x)$ be a sequence of functions of V such that $p_n(0)=0$ and $\sum_1^\infty V(p_n)<\infty$. Then $\sum_1^\infty p_n(x)$ is absolutely and uniformly convergent to a function $p(x)\varepsilon V$ and

$$(1) \qquad \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} g(x)dp_n(x)=\int_{-\infty}^{\infty} g(x)dp(x).$$

The assertion concerning $p(x)$ is readily proved. After proving this the author proves (1) under the conditions that $g(x)$ is a bounded piecewise absolutely continuous function in $(-\infty, \infty)$. He then successively removes these extra conditions. After completing the proof of this theorem, the author proves the following result. Let $g(x)$ be a given Borel measurable function defined everywhere in $(-\infty, \infty)$, with the property that $\int_{-\infty}^{\infty} g(x)dp(x)$ exists whenever $p(x)\varepsilon V$. Then $g(x)$ is bounded everywhere in $(-\infty, \infty)$.

W. T. Martin (Syracuse, N. Y.).

Theory of Series

González, Mario O. On the calculation of the radius of convergence of power series. Revista Soc. Cubana Ci. Fís. Mat. 1, 84–87 (1943). (Spanish) [MF 8940]

Given a power series, form two new power series by taking alternate terms. The author observes that the radius of convergence of the original series is the square root of the smaller of the radii of convergence of the two new series.

R. P. Boas, Jr. (Cambridge, Mass.).

Pollard, Harry. Subseries of a convergent series. Bull. Amer. Math. Soc. 49, 730–731 (1943). [MF 9312]

The subseries of a given series $u_1+u_2+\cdots$ are associated with points x of the interval $0\leqq x\leqq 1$ by using the Rademacher functions $R_n(x)$ to define

$$(1) \qquad \varphi(x)=\sum_{n=1}^{\infty}\tfrac{1}{2}[1+R_n(x)]u_n.$$

Hill [Bull. Amer. Math. Soc. 48, 103–108 (1942); these Rev. 3, 147] showed, without use of Rademacher functions, that, if $\sum|u_n|$ converges and $\sum u_n=s$, then

$$(2) \qquad \int_0^1 \varphi(x)dx=\tfrac{1}{2}s.$$

It is shown here that (1) converges on a set of positive measure if and only if both $\sum u_n$ and $\sum u_n^2$ converge. If both converge, then (1) converges almost everywhere and (2) holds. In a written communication to the author, Ulam observed that, if $\sum|u_n|<\infty$ and $u_n\neq 0$ for an infinite set of n, then the set E of values attained by $\varphi(x)$ is perfect. Ulam asked for characterization of such perfect sets E. H. Hornich [Monatsh. Math. Phys. 49, 316–320 (1941); these Rev. 3, 294] proved that E is perfect and gave further facts about E.

R. P. Agnew (Ithaca, N. Y.).

Goddard, L. S. A problem in the summation of series. Proc. Cambridge Philos. Soc. 39, 200–202 (1943). [MF 9202]

By use of partial fractions two infinite series that occur in electromagnetic theory are expressed in finite form. The results are

$$\alpha^2n^2\sum_{m=1}^{\infty}\frac{\sin^2(\pi m/\alpha)}{m(m^2-\alpha^2n^2)}=-\sum_{m=1}^{\alpha n-1}\frac{\sin^2(\pi m/\alpha)}{m},$$

$$\alpha^2n^2\sum_{m=1}^{\infty}\frac{m\sin^2(\pi m/\alpha)}{(m^2-\alpha^2n^2)^2}=\frac{\pi^2n}{4\alpha}+\tfrac{1}{2}\alpha n\sum_{m=1}^{\alpha n-1}\frac{\sin^2(\pi m/\alpha)}{m^2},$$

where n and α are positive integers, $\alpha>1$.

J. D. Hill.

Obreschkoff, Nikola. Über das Riemannsche Summierungsverfahren. Math. Z. 48, 441–454 (1942). [MF 8720]

The series $a_0+a_1+\cdots$ is said to be summable (R, p) (p a positive integer) to sum s if the series

$$a_0+\sum_{n=1}^{\infty}a_n((\sin nh)/nh)^p$$

converges in the neighborhood of $h=0$, and its sum tends to s as $h\to 0$. It is well known that, if $p>\alpha$, summability (C, α) does not necessarily imply summability (R, p). The author shows that (1) if the series is summable (C, α) and if

$$n^{-1}\sum_{\nu=0}^{n}|s_\nu^{\alpha-1}|=O(1),$$

$s_\nu^{\alpha-1}$ Cesàro means, the series is summable (R, p). Moreover, (2) if the series is summable $|C, p|$, it is also summable (R, p). (3) A result analogous to (1) holds for a modified method of Riemann which defines the limit of a sequence $\{b_\nu\}$ as

$$C_p\lim_{h\to 0}h\sum_{n=1}^{\infty}b_n((\sin nh)/nh)^p,$$

where C_p is a normalizing constant.

A. Zygmund.

Chandrasekharan, K. The second theorem of consistency for absolutely summable series. J. Indian Math. Soc. (N.S.) 6, 168–180 (1942). [MF 8376]

The theorem states that summability of a series $|\lambda, k|$ (absolute Riesz summability of order k for a sequence $\{\lambda_n\}$) implies its summability $|\mu, k|$ to the same sum if μ is a logarithmico-exponential function of λ such that $\mu=O(\lambda^\Delta)$ for some constant Δ.

R. P. Boas, Jr.

Chandra Sekharan, K. Bessel summation of series. Proc. Indian Acad. Sci., Sect. A. 17, 219–229 (1943). [MF 9267]

This paper and the one reviewed below discuss a summation method based on Bessel functions [S. Minakshisundaram, Madras thesis] and the corresponding absolute summation method. Defining

$$\alpha_\mu(t)=2^\mu\Gamma(\mu+1)J_\mu(t)t^{-\mu},$$

set $\varphi_\mu(t)=\sum_0^\infty a_n\alpha_\mu(nt)$ (assuming that the series converges). If $\varphi_\mu(t)\to s$ as $t\to 0$, the series $\sum a_n$ is summable J_μ to s; if $\varphi_\mu(t)$ is of bounded variation in an interval containing the origin, $\sum a_n$ is summable $|J_\mu|$. Summability J_1 is Lebesgue's method; a more general method, in which $\alpha_\mu(t)$ is raised to a higher power than the first, includes Riemann summability as a special case. If $\sum a_n$ is summable (C, r) or $|C, r|$, it is summable J_μ or $|J_\mu|$ for $\mu>r+\frac{1}{2}$; if $\sum a_n$ is summable

J_μ or $|J_\mu|$, and some additional conditions are satisfied, then it is summable (C, r) or $|C, r|$ for $r>\mu+\frac{1}{2}$.

R. P. Boas, Jr. (Cambridge, Mass.).

Chandrasekharan, K. The absolute Bessel-summability of series. Bull. Calcutta Math. Soc. **34**, 187–196 (1942). [MF 9001]

An absolute summation method is defined which corresponds to the Bessel summation method used in the paper reviewed above. Let $\varphi_\mu(t)=\sum_0^n a_n\alpha_\mu(nt)$, assuming that the series converges. If $\varphi_\mu(t)$ is of bounded variation in an interval containing the origin, $\sum a_n$ is said to be summable $|J_\mu|$. The author discusses relations between $|J_\mu|$ summability for different μ's, and between $|J_\mu|$ summability and absolute Cesàro summability.

R. P. Boas, Jr. (Cambridge, Mass.).

Chandrasekharan, K. Bessel-summability of the product of two series. J. Indian Math. Soc. (N.S.) **7**, 31–35 (1943). [MF 9378]

The author makes the following definition: the Bessel-product of two series $\sum a_\mu$ and $\sum b_\nu$ is equal to $\sum c_n$, where

$$c_n=\sum_{n-(\mu^2+\nu^2)^{\frac{1}{2}}}a_\mu b_\nu.$$

[It is apparently intended that c_n should be defined not only for positive integral n, but for all values of n of the form $(\mu^2+\nu^2)^{\frac{1}{2}}$, and that the terms of the product series should be arranged in the order of increasing n.] Let

$$\alpha_\mu(t)=2^\mu\Gamma(\mu+1)t^{-\mu}J_\mu(t),$$

where $J_\mu(t)$ is the Bessel function of order μ. A series $\sum a_n$ is by definition summable J_μ to the sum s if $\sum a_n\alpha_\mu(nt)$ converges for $t>0$ and if

$$\lim_{t\to 0}\sum a_n\alpha_\mu(nt)=s.$$

The author also specifies that in what follows we assume that $\sum a_n\alpha_\mu(nt)$ converges in such a manner as to permit the integration of the series term by term if we are given that $\sum a_n$ is summable J_μ. A series $\sum a_n$ is said to be absolutely summable J_μ or summable $|J_\mu|$ if $\sum a_n\alpha_\mu(nt)$ converges for $t>0$ and represents a function of t of bounded variation in some finite interval including the origin. In the following theorems concerning the Bessel-product $\sum c_r$ of $\sum a_n$ and $\sum b_n$ the constants μ and ν are each positive. (1) If $\sum a_n$ is summable J_μ and $\sum b_n$ is summable J_ν then $\sum c_r$ is summable $J_{\mu+\nu+1}$. (2) If $\sum a_n$ is summable $|J_\mu|$ and $\sum b_n$ is summable J_ν, then $\sum c_r$ is summable $J_{\mu+\nu}$. (3) If $\sum a_n$ is summable $|J_\mu|$ and $\sum b_n$ is summable $|J_\nu|$, then $\sum c_r$ is summable $|J_{\mu+\nu+1}|$. *F. W. Perkins.*

Fuchs, W. H. J. and Rogosinski, W. W. On typical means. Quart. J. Math., Oxford Ser. **14**, 27–48 (1943). [MF 8628]

In this paper the authors apply the theory of continuous Hausdorff methods of summation, already studied by Rogosinski [Proc. Cambridge Philos. Soc. **38**, 344–363 (1942); these Rev. **4**, 195], to the study of summation by typical means. Continuous Hausdorff transforms are of the type $t(x)=\int_0^1 s(xt)d\phi(t)$; the integral is a Lebesgue-Stieltjes integral and the restrictions imposed on $\phi(t)$ are those encountered in any study of Hausdorff transforms. Typical means $T^{(\lambda)}$ are obtained by setting $s(x)=s^{(\lambda)}(x)=s_n$, $\lambda_n\leqq x<\lambda_{n+1}$, to get the transform

$$t^{(\lambda)}(x)=\sum_{\lambda_k\leqq x}s_k\{\phi(\lambda_{k+1}/x-0)-\phi(\lambda_k/x-0)\}.$$

The main problem of the paper is a comparison of the strength of various methods of summation $T^{(\lambda)}$ for a given λ-sequence. As in the theory of continuous Hausdorff transforms the zeros of the Mellin transform of $\phi(t)$, $T(z)=\int_+^1 t^z d\phi(t)$, $\Re(z)\geqq 0$, play a decisive rôle.

In §1 of the paper it is proved that a necessary condition for regularity of the typical means is that $\lambda_{n+1}/\lambda_n\to 1$ and that $\phi(t)$ is of bounded variation; it is sufficient for regularity that $\phi(t)$ is of bounded variation. In §2 some general theorems involving the methods $T^{(\lambda)}$ are given. In this connection it is of interest that, if $T(0)=0$, then $T^{(\lambda)}\supset C^{(\lambda)}$ (Riesz means) for every λ-sequence, while this is not true for any other zero of $T(z)$. In §3, the main part of the paper, a detailed study is made of the special methods of summation

$$U_{p,q}=H_p^{(\lambda)}+\alpha_1 H_{p+1}^{(\lambda)}+\cdots+\alpha_q H_{p+q}^{(\lambda)},\qquad \alpha_q\neq 0,$$

where p and q are nonnegative integers and where the H_k are the Hölder methods of order k. *H. L. Garabedian.*

Mears, Florence M. The inverse Nörlund mean. Ann. of Math. (2) **44**, 401–410 (1943). [MF 8870]

For a general discussion of the Nörlund mean the reader should consult C. N. Moore's "Summable Series and Convergence Factors" [Amer. Math. Soc. Colloquium Publ., v. 22, New York, 1938]. In the paper under review necessary definitions are given and twenty theorems proved. These concern both summability and absolute summability and a particular treatment is given of the inverse of Cesàro summability. To the reviewer the more interesting theorems are those relating to the Cauchy product of two series. The paper registers a genuine advance in this field. However, for a statement of the theorems reference must be made to the paper itself. *T. Fort* (Bethlehem, Pa.).

Neder, Ludwig. Eine notwendige und hinreichende Bedingung für Doppelreihen. Math. Z. **48**, 497–499 (1942). [MF 8723]

The following theorem is proved: a necessary and sufficient condition that the double series $\sum a_{\mu,\nu}$, together with all of its rows and columns, be convergent is that the partial sums $s_{m,n}=\sum_{\mu=1}^m\sum_{\nu=1}^n a_{\mu,\nu}$ have a development of the form $s_{m,n}=s-\varphi_m-\psi_n+\chi_{m,n}$, where $\varphi_m\to 0$ as $m\to\infty$, $\psi_n\to 0$ as $n\to\infty$ and $\chi_{m,n}\to 0$ as $m+n\to\infty$.

H. L. Garabedian (Evanston, Ill.).

Lösch, Friedrich. Über die restringierte Limitierung von Doppelfolgen nach den Verfahren von Cesàro, Hölder, und Euler-Knopp. Math. Z. **48**, 105–127 (1942). [MF 8564]

Let $\{A, B\}$ denote the double-sequence transformation

$$S_{mn}=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}a_{m\mu}b_{n\nu}s_{\mu\nu}$$

from $s_{\mu\nu}$ to S_{mn} determined by matrices A and B of real or complex constants $a_{m\mu}$ and $b_{n\nu}$. It is shown that, if A_r is one of the matrices C_r, H_r, E_r of Cesàro, Hölder and Euler, respectively, and if r and s are nonnegative integers, then a restrictedly convergent sequence $s_{\mu\nu}$ is summable $\{A_r, A_s\}$ if and only if it is restrictedly bounded $\{A_r, A_s\}$; and that, if $s_{\mu\nu}$ is restrictedly convergent to L and restrictedly bounded $\{A_r, A_s\}$, then it is restrictedly summable $\{A_r, A_s\}$ to L. If r and s are nonnegative integers, a sequence $s_{\mu\nu}$ restrictedly summable $\{C_r, C_s\}$ is restrictedly summable

$\{H_r, H_s\}$ if and only if it is restrictedly bounded $\{H_r, H_s\}$. When $s_{\mu\nu}$ is restrictedly summable $\{C_r, C_s\}$ and $\{H_r, H_s\}$ to L_1 and L_2, respectively, then $L_1 = L_2$. *R. P. Agnew.*

Meyer-König, W. Die Umkehrung des Euler-Knoppschen und des Borelschen Limitierungsverfahrens auf Grund einer Lückenbedingung. Math. Z. 49, 151–160 (1943). [MF 9194]

The following Tauberian gap theorem is proved. Let $\theta > 0$, let $n_1, n_2, \cdots$ be a sequence of indices for which $n_{k+1} > (1+\theta)n_k$ for all sufficiently great k and let $\sum a_n$ be a series for which $a_n = 0$ when $n \neq n_1, n_2, \cdots$. Let $a_n = O(\alpha^n)$ for some positive number α, and let

$$b(x) = e^{-x} \sum_{n=0}^{\infty} (x^n/n!) s_n,$$

where $s_n = a_0 + \cdots + a_n$, denote the Borel exponential transform of $\sum a_n$. If $b(x) \to s$ as $x \to \infty$ (if $\sum a_n$ is summable B to s) then $\sum a_n$ converges to s. Since summability by an Euler-Knopp method E_p implies both summability B and an order condition $a_n = O(\alpha^n)$, the above theorem implies a similar Tauberian gap theorem for E_p summability in which the order condition $a_n = O(\alpha^n)$ is removed from the hypothesis. *R. P. Agnew* (Ithaca, N. Y.).

Hardy, G. H. An inequality for Hausdorff means. J. London Math. Soc. 18, 46–50 (1943). [MF 9212]

Let $\{b_n\}$ be the transform of $\{a_n\}$ by the Hausdorff transformation

$$b_n = \int_0^1 \left\{ \sum_{m=0}^{n} \binom{n}{m} (1-t)^{n-m} t^m a_m \right\} d\chi(t),$$

in which $\chi(t)$ is monotone increasing with t, $\chi(0) = \chi(0+) = 0$ and $\chi(1) = 1$. If $a_n \geqq 0$ and $p > 1$, then

$$\sum b_n^p < \left(\int_0^1 t^{-1/p} d\chi(t) \right)^p \sum a_n^p$$

unless $a_n = 0$ for all n or the transformation is the identity. The multiplier of $\sum a_n^p$ is the best possible.

R. P. Agnew (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Szegö, G. On conjugate trigonometric polynomials. Amer. J. Math. 65, 532–536 (1943). [MF 9384]

Let

$$u(\theta) = \tfrac{1}{2}a_0 + \sum_1^n (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta),$$

$$v(\theta) = \sum_1^n (a_\nu \sin \nu\theta - b_\nu \cos \nu\theta),$$

where the coefficients a_ν and b_ν are real. If $|u(\theta)| \leqq 1$ for all θ, then $|v(\theta)|$ does not exceed

$$M_n = 2(n+1)^{-1} \sum_{\substack{1 \leqq m \leqq n \\ m \text{ odd}}} \cot\,(m\pi/(2(n+1))).$$

This M_n cannot be replaced by any smaller number.

A. Zygmund (South Hadley, Mass.).

Hardy, G. H. and Rogosinski, W. W. Notes on Fourier series. I. On sine series with positive coefficients. J. London Math. Soc. 18, 50–57 (1943). [MF 9213]

Short proofs are given of several known results and one new one. Let $f(\theta)$ be odd and integrable and $S(\theta) = \sum b_n \sin n\theta$ its Fourier series. By (1.1) $f(\theta) \to \frac{1}{2}\pi A(C, \alpha)$, where $\alpha > 0$, $\theta \to 0^+$, is meant

$$(1.2) \qquad (\alpha/\theta^\alpha) \int_0^\theta (\theta - t)^{\alpha-1} f(t)\,dt \to \tfrac{1}{2}\pi A.$$

By (1.3) $nb_n \to A(C, \beta)$, where $\beta > 0$, it is meant that the βth order Cesàro mean of nb_n tends to A. Special cases ($\alpha = \beta = 1$) are (1.4) $(1/\theta)\sum (b_n/n)(1 - \cos n\theta) \to \frac{1}{2}\pi A$ and (1.5) $n^{-1}(b_1 + 2b_2 + \cdots + nb_n) \to A$, respectively. A further condition of importance is (1.6) $nb_n \geqq -1$. The following results are proved. (I) [Chaundy and Jolliffe] If $\{b_n\}$ is positive and decreasing, then $nb_n \to 0$ is a necessary and sufficient condition (n.s.c.) for either the uniform convergence of S or the continuity of f. (II) [Hardy] If $\{b_n\}$ is positive and decreasing and $A > 0$ then $nb_n \to A$ is a n.s.c. that $f(\theta) \to \frac{1}{2}\pi A$. (III)[Szász] In the presence of (1.6), (1.5) and (1.4) are equivalent. (IV) [Paley] If $\{b_n\}$ satisfies (1.6) and f is continuous, then S is uniformly convergent. (V) (new) If $\{b_n\}$ satisfies (1.6), then (1.3) for any $\beta \geqq 1$ is a n.s.c. for (1.1) for any $\alpha > 0$. *H. Pollard.*

Iyengar, K. S. K. A Tauberian theorem and its application to convergence of Fourier series. Proc. Indian Acad. Sci., Sect. A. 18, 81–87 (1943). [MF 9329]

The author defines as the summation method (N, k) the Nörlund mean based on the coefficients $b_{k,n}$ of

$$\{\textstyle\sum_{j=0}^{\infty} x^j/(j+1)\}^k.$$

The Tauberian theorem is that, if $\{S_n\}$ is summable (N, k) (k a positive integer) and $S_n - S_{n-1} \leqq An^{-\mu}$, $A > 0$, $0 < \mu < 1$, then $\{S_n\}$ converges. The author proves further that the Fourier series of an integrable $f(x)$ is summable $(N, 1)$ to $f(x)$ if $[f(x+h) - f(x)] \log |h| \to 0$ as $h \to 0$. The two theorems together imply a convergence criterion of Hardy and Littlewood [see Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 35]. *R. P. Boas, Jr.*

Iyengar, K. S. K. New convergence and summability tests for Fourier series. Proc. Indian Acad. Sci., Sect. A. 18, 113–120 (1943). [MF 9331]

Let $f(x)$ be integrable, $\psi_x(t) = |f(x+t) + f(x-t) - 2f(x)|$. A well-known theorem [Zygmund, Trigonometrical Series, Warsaw-Lwów, 1935, p. 49] states that the Fourier series of $f(x)$ is summable (C, r), $r > 0$, to $f(x)$ if (*) $\int_0^t \psi_x(u)\,du = o(t)$ $(t \to 0)$. The author extends this by associating with a class of Nörlund means a corresponding class of continuity conditions such that the series is summable by the appropriate mean if the corresponding continuity condition is satisfied. The simplest form of the theorem reads as follows. Let $\{b_n\}$ be a monotonic nondecreasing positive sequence, $\sum b_n$ divergent, $B_n = \sum_{\nu=0}^{n} b_\nu$, $B_x = B_{[x]}$, $S_n(x)$ the nth partial sum of the Fourier series of $f(x)$; then $\sum_{i=0}^{n} b_i s_{n-i}(x)/B_n \to f(x)$ provided that (**) $\int_0^t \psi_x(1/u)(B_u/u)\,du = o(B_t)$ as $t \to \infty$. If $b_n = n^{-\sigma}$, $0 \leqq \sigma < 1$, this is equivalent to the theorem quoted at the beginning of the review. However, if $\sum_{i=1}^{n} B_i/i \neq O(B_n)$, the summation process is weaker than any (C, r), $r > 0$, while the continuity condition (**) is stronger than (*). The author states a Tauberian theorem for the summation process, which allows him to generalize the Hardy-Littlewood convergence criterion [cf. the preceding review, where

the case $b_n=1/n$ is considered] by weakening the condition on the coefficients and strengthening the continuity condition. *R. P. Boas, Jr.* (Cambridge, Mass.).

Bellman, Richard. Random summability and Fourier series. Bull. Amer. Math. Soc. **49**, 732–733 (1943). [MF 9313]

Let $r_1(t), r_2(t), \cdots, r_k(t)=\text{sign sin } 2^{k+1}\pi t, \cdots$ be Rademacher's functions. A series with partial sums s_k is said to be random summable to sum s if

$$\lim_{n\to\infty}\frac{\sum_1^n s_k(1+r_k(t))}{\sum_1^n (1+r_k(t))}=s$$

for almost every t. If $f\in L$, the Fourier series of f is random summable to $f(x)$ for almost every x. *A. Zygmund.*

Ríos, Sixto. On the rearrangement of series of functions and its applications. Revista Mat. Hisp.-Amer. (4) **3**, 100–128 (1943). (Spanish) [MF 8950]

The author shows that series $\sum a_n e^{-\lambda_n s}$ in which the λ_n tend to $+\infty$ but not monotonically (λ_n real) and which are such that $\sum|e^{-\lambda_n s}-e^{-\lambda_{n+1}s}|$ converges uniformly in each bounded region in $\Re(s)>0$ have properties analogous to those of ordinary Dirichlet series. *S. Mandelbrojt.*

Bückner, Hans. Über die Darstellung einer Funktion durch ihre iterierten arithmetischen Mittel. Jber. Deutsch. Math. Verein. **52**, 200–204 (1942). [MF 9053]

Let $f_0(x)$, continuous for $0\leqq x\leqq 1$, have arithmetic mean $f_1(x)=(1/x)\int_0^x f_0(s)ds$, $f_1(0)=f_0(0)$; form the sequence of iterated means $f_1(x), f_2(x), \cdots$, with $f_{n+1}(x)$ the arithmetic mean of $f_n(x)$; and let

$$M_n(f_0)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}f_k.$$

Then

$$M_n(f_0)=f_0-\sum_{k=0}^{n-1}M_k(f_1).$$

It is shown that

$$\int_0^x[f_0(s)-f_1(x)]^2ds=\int_0^x[f_0(s)-f_1(s)]^2ds,$$

and from this the principal identity is developed:

$$\int_0^x f_0^2(s)ds=x\sum_{k=0}^{n}M_k^2(f_1)+\int_0^x M_{n+1}^2(f_0)ds.$$

It follows that $x\sum_{k=0}^{\infty}M_k^2(f_1)$ converges, and, since

$$\lim_{k\to\infty}M_k(f_1)=0,\quad \lim_{k\to\infty}\int_0^x M_k^2(f_1)ds=0,$$

that

$$f_1(x)=\sum_{k=0}^{\infty}M_k(f_2),\quad \int_0^x f_1^2(s)ds=x\sum_{k=0}^{\infty}M_k^2(f_2).$$

E. F. Beckenbach (Austin, Tex.).

Differential Equations

Chapman, H. Wallis. Intergrating factors of ordinary differential equations of the second order: a geometrical interpretation. Math. Gaz. **27**, 159–165 (1943). [MF 9527]

Wylie, C. R., Jr. On the forced vibrations of non-linear springs. J. Franklin Inst. **236**, 273–284 (1943). [MF 9059]

Approximate solutions, of period 2π, of the Duffing equation

$$\lambda^2y''+y+ay^3=\cos x$$

are obtained by several different methods and compared with the solution obtained by numerical methods where the error is known to be small. The approximate solution of the form $A\cos x$, obtained by choosing A as an optimum constant, is not satisfactory. The solution of the form $A\cos x+B\cos 3x$, where A and B are determined in the usual way, is shown to be very satisfactory. Other procedures are also shown to be satisfactory. The results are carried out over a considerable range of values of the dimensionless parameters a and λ. *N. Levinson.*

Levinson, Norman. On certain non-linear differential equations of the second order. Proc. Nat. Acad. Sci. U. S. A. **29**, 222–223 (1943). [MF 9022]

A number of results of great interest concerning the qualitative character in the large of the solutions of a class C of ordinary differential equations of second order are announced. The class C is not defined, but it is stated that it includes differential equations of importance in applied mathematics. Included in the class C, in particular, are the differential equations

$$(*)\qquad \frac{d^2x}{dt^2}+f(x)\frac{dx}{dt}+g(x)=e(t),$$

in which $e(t)$ is a periodic function of least period L, $f(x)$ and $g(x)$ are analytic functions which behave appropriately for large values of x and $f(x)$ is positive. It is known that any differential equation of class C possesses at least one periodic solution of period L.

The differential equations of class C can be classified according to the behavior of their solutions, when $t\to\infty$, into four mutually exclusive subclasses which include all cases. A number of interesting properties of the solutions in each of the classes is announced. For example, one of the subclasses is characterized by the property that the differential equations in it possess at least one solution of period qL, q an integer larger than one; in other words, what is commonly called a subharmonic solution exists. If this is the case, then the following conclusions (among others) can be drawn: (a) no subharmonic solutions with period distinct from qL exist; (b) all solutions, other than unstable subharmonic ones, tend to the stable subharmonic ones or to the solution of period L as $t\to+\infty$. *J. J. Stoker.*

Levinson, Norman. Correction to "On certain non-linear differential equations of the second order." Proc. Nat. Acad. Sci. U. S. A. **29**, 281 (1943). [MF 9023]

The author withdraws the statement that the differential equation (*) [see the preceding review] is in the class C if $f(x)$ is positive. *J. J. Stoker* (New York, N. Y.).

Levinson, Norman. On the existence of periodic solutions for second order differential equations with a forcing term. J. Math. Phys. Mass. Inst. Tech. **22**, 41–48 (1943). [MF 8942]

The existence of a periodic solution of period L of the differential equation

$$(*)\qquad \frac{d^2x}{dt^2}+f\left(x,\frac{dx}{dt}\right)\frac{dx}{dt}+g(x)=e(t)$$

is proved. The function $e(t)$ is assumed to be periodic of period L. The assumptions with regard to the "damping coefficient" $f(x, dx/dt)$ and the "restoring force" $g(x)$ are general enough that the theorem holds in the great majority of the cases arising in practice. The existence of a periodic solution of (*) is a consequence of the following theorem. There exists a simple closed curve C in an (x, v) plane such that any curve $x=x(t)$, $v=dx(t)/dt$ with $x(t)$ a solution of (*) intersects C, with increasing t, by crossing from the exterior to the interior domain bounded by C. Most of the paper is devoted to a proof of this theorem.

J. J. Stoker (New York, N. Y.).

Vessiot, E. Sur les équations aux dérivées partielles du second ordre, $F(x, y, z, p, q, r, s, t)=0$, intégrables par la méthode de Darboux. J. Math. Pures Appl. (9) **21**, 1–66 (1942). [MF 9296]

This is a continuation of a paper under the same title in the same J. (9) **18**, 1–61 (1939). If the equation

$$F(x, y, z, p, q, r, s, t)=0$$

is put in the parametric form

$$r=\rho(x, y, z, p, q, a, b), \quad s=\sigma(x, y, z, p, q, a, b),$$
$$t=\tau(x, y, z, p, q, a, b),$$

it has previously been shown [Bull. Soc. Math. France **52**, 336–395 (1924)] that it can be associated with a pencil (faisceaux) of infinitesimal transformations having the base:

$$X_1f\equiv\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}+\rho\frac{\partial f}{\partial p}+\sigma\frac{\partial f}{\partial q}, \quad X_2f\equiv\frac{\partial f}{\partial a},$$

$$X_3f\equiv\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial z}+\sigma\frac{\partial f}{\partial p}+\tau\frac{\partial f}{\partial q}, \quad X_4f\equiv\frac{\partial f}{\partial b}.$$

The author showed [part 1 of the present paper] for equations $F=0$, having distinct characteristic systems, which are integrable by Darboux's method, that each singular subpencil of the associated pencil of transformations X_θ has at least two invariants. An extensive study is made of the pencils of transformations in those cases where the invariants are of the first and second order. In the part of the paper under review the author gives a complete table of the types to which one can reduce, by contact transformations, all those equations $F=0$ which have at least two invariants of first or second order for each system of characteristics and for which the general integral is explicit. Under these restrictions $F=0$ can be reduced by a contact transformation to the form $s=f(x, y, z, p, q)$. A comparison with Goursat's [Ann. Fac. Sci. Univ. Toulouse (2)1, 439–464 (1899)] classification of the equation $s=f(x, y, z, p, q)$ is made.

F. G. Dressel (Durham, N. C.).

Grünberg, G. A. Some theorems relating to the evaluation of potentials and charges induced on conductors placed in a given external electric field. C. R. (Doklady) Acad. Sci. URSS (N.S.) **38**, 203–205 (1943). [MF 9244]

It is shown that the potentials and charges indicated in the title may be found by quadratures in terms of the density distributions on the conductors in the absence of external fields and when the conductor under consideration has unit charge on it while all the other conductors of the system are not charged.

H. Poritsky.

Magnus, Wilhelm. Über Eindeutigkeitsfragen bei einer Randwertaufgabe von $\Delta u + k^2u = 0$. Jber. Deutsch. Math. Verein. **52**, 177–188 (1942). [MF 9050]

The writer proves some uniqueness theorems of an unusual sort for solutions of the equation $\Delta u+k^2u=0$ which are important in the theory of the reflection and bending of waves. The first of these is the following. Let F be a sufficiently smooth closed surface (the hypotheses are not stated completely) in (x, y, z) space and let u and its normal derivative be continuous on and outside F and suppose u is of class C'' outside F where it is a solution of our equation. Suppose also that u vanishes on F, that $|ru|$ is bounded outside F and

$$\lim_{r\to\infty} r(\partial u/\partial r+iku)=0, \quad r=(x^2+y^2+z^2)^{\frac{1}{2}}.$$

Then $u\equiv 0$ outside F.

Let F be a doubly counted (that is, as though it were considered as the limit of a sequence of closed surfaces) region in the (x, y) plane bounded by a smooth curve. Let Π be a vector of class C'' everywhere (except possibly at the bounding curve and possibly having different values on the opposite sides of F), each of whose components satisfies all the conditions of the u above except that of vanishing on F. Suppose also that Π is representable in the form

$$\Pi=\iint_F R^{-1}\exp(-ikR)\cdot s\,dF, \tag{1}$$

$$R^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2,$$

for some vector $s=(s_1, s_2, s_3)$ (satisfying the differentiability conditions on ϵ below) and that the vector

$$\epsilon=(-ik)^{-1}[k^2\Pi+\text{grad div }\Pi]$$

remains bounded near F, its x and y components being of class C' in x and y interior to F. The second result is: the (x, y) divergence $(s_{1x}+s_{2y})$ and the component of s in the (x, y) plane normal to the bounding curve F^* of F must vanish continuously on F^*. This edge condition is generalized to any surface F in 3-space which is smooth except for a finite number of smooth boundaries and edges.

The determination of the vector s in (1) for the plane region F from prescribed x and y components of ϵ may be reduced to two integral equations of the form

$$E(x, y)=\iint_F R^{-1}\exp(-ikR)g(x_0, y_0)dx_0dy_0,$$

$$R^2=(x-x_0)^2+(y-y_0)^2.$$

The third result is: if $E\equiv 0$ on F, then $g\equiv 0$ on F if g is assumed continuous.

C. B. Morrey, Jr.

Schröder, Kurt. Zur Theorie der Randwertaufgaben der Differentialgleichung $\Delta\Delta U=0$. Math. Z. **48**, 553–675 (1943). [MF 9006]

The partial differential equation $\Delta\Delta u=0$ of fourth order is of prime importance in the theory of elasticity. In this paper the author proves existence and uniqueness theorems for the interior and exterior problems in three dimensions and in the plane. The solution of the exterior problem for space is new; the interior problem in space and the problems in the plane are included to give more general results than existed before, to fill certain gaps in previous proofs and to present a unified approach to the whole set of problems. In the three dimensional case, the following hypotheses are made. (1) The region R is simply connected and bounded

by a finite surface S; S has a tangent plane at each point. If $z=\varphi(x, y)$ is the equation of S with respect to local Cartesian coordinates set up in the tangent plane with origin at the point of tangency, then $\partial\varphi/\partial x$ and $\partial\varphi/\partial y$ satisfy a uniform Hölder condition over a fixed circle $x^2+y^2<R^2$. (2) When a boundary function $f(p)$, p on S, is employed, it is assumed continuous and to have continuous partial derivatives with respect to the local Cartesian coordinates, also to satisfy a uniform Hölder condition and to be the values assumed on S by a function in R which is twice differentiable in R.

One of the two principal theorems is as follows. If R is a bounded domain whose boundary S satisfies the conditions (1), and if ρ, σ, τ are three functions defined on S satisfying the conditions (2), then there exists exactly one function U in R satisfying $\Delta\Delta U=0$ and whose first partial derivatives are continuous in $R+S$ and assume the values ρ, σ and τ, respectively. The exterior problem is similarly stated and solved. In this case the solution function is required to vanish like $1/r$ at ∞.

The method used is that of integral equations. The greater part of the paper is concerned with obtaining the necessary integral identities and geometrical inequalities related to the surface which are required for the proof of the existence of the solution of the integral equations used.

J. W. Green (Aberdeen, Md.).

Schröder, K. Über die Ableitungen biharmonischer Funktionen am Rande. Math. Z. **49**, 110–147 (1943). [MF 9192]

The behavior of the derivatives of a harmonic function on the boundary S of a region R in which the harmonic function is determined by boundary values has been studied by O. D. Kellogg. In this paper the author obtains similar results for biharmonic functions, that is, functions satisfying $\Delta\Delta U=0$. The proofs are carried out by suitable modifications of Kellogg's methods. Although a number of different results are obtained, the following is the principal one. Consider a point P_0 of the bounding surface S which has a tangent plane; P_0 is taken as the origin of Cartesian coordinates in the plane, and it is supposed that, near P_0, S can be expressed as $z=\varphi(x, y)$. It is assumed that $\varphi(x, y)$ has continuous partial derivatives of nth order and that each of these derivatives satisfies a sort of Hölder condition near P_0 except that the $|x_1-x_2|$ of the Hölder condition is replaced by $D(x_1-x_2)$, where $\int_0^{+0}(D(t)/t)dt$ converges. Concerning the biharmonic function U, it is assumed that U is continuous in $R+S$ together with its first partial derivatives. The boundary values of the first derivatives are assumed to have continuous partial derivatives of order $n-1$ satisfying the modified Hölder condition mentioned above. Under these assumptions it is proved that any nth derivative of U tends to a limit as the boundary is approached along the normal, and that, with this limit as the definition of this nth derivative on the boundary, the derivative is continuous on S for any method of approach.

J. W. Green (Aberdeen, Md.).

Brown, Herbert Kapfel. The resolution of boundary value problems by means of the finite Fourier transformation: general vibration of a string. J. Appl. Phys. **14**, 609–618 (1943).

Let $V(x, t)$ satisfy the partial differential equation

$$V_{tt}=C_1(t)V_{xx}-C_2(t)V+P(x, t),\quad 0<x<\pi;\ t>0,$$

and the boundary conditions $V(0, t)=C_3(t)$, $V(\pi, t)=C_4(t)$, $V(x, 0)=F_1(x)$, $V_t(x, 0)=F_2(x)$, where the C's, F's and P are prescribed functions. The author finds the function V in terms of these prescribed functions and two functions $V_1(x, t)$, $V_2(x, t)$ which are solutions of special cases of the above boundary value problem. The function V_1 is the solution of the above problem when P, C_3, C_4 and F_1 are identically zero and $F_2=1-x/\pi$, while V_2 is the solution when P, C_3, C_4 and F_2 are identically zero and $F_1=1-x/\pi$.

R. V. Churchill (Ann Arbor, Mich.).

Pinl, M. Zur Theorie der kompressiblen Potentialströmungen. III. Charakteristiken, Bicharakteristiken und Eikonal der linearisierten Grundgleichung. Z. Angew. Math. Mech. **22**, 305–311 (1942). [MF 9118]

[Parts I and II appeared, as joint papers by Behrbohm and Pinl, in the same Z. **21**, 193–203, 341–350 (1941); cf. these Rev. **4**, 159.] The well-known nonlinear differential equation in the velocity potential for two dimensional steady incompressible flow is associated with the linear equation

$$(1)\qquad U_{\alpha\alpha}+U_{\beta\beta}+(\gamma^3/((\mu-1)(\alpha^2+\beta^2)+\lambda\gamma^3))U_{\gamma\gamma}=0,$$

where μ, λ are constants and U is a generalized Minkowski function of support. It is of interest to note in passing that the correlated metric for (1) is not conformally Euclidean. By suitable change of independent variable (1) is replaced by an equation whose coefficients contain a single independent variable. The associated bicharacteristic equation (and the eikonal equation also) can now be integrated by quadratures. Accordingly the characteristic manifolds are immediately obtainable, but the writers actually integrate the characteristic equation derived from the transformed (1) to get these manifolds. As it stands the paper amounts to the application of standard classical procedure to the specific equation (1). It would seem of interest actually to evaluate the integrals arising out of the "quadratures." From the form of the integrands the reviewer would expect that such evaluation could be made in terms of elliptic or hyperelliptic functions.

D. G. Bourgin (Urbana, Ill.).

Buhl, Adolphe. Ondes. Analyse. Géométrie. Arithmétique. J. Math. Pures Appl. (9) **21**, 123–139 (1942). [MF 9300]

This paper deals largely with propagation with conservation of area. The basic partial differential equation is $z_x^2+z_y^2=\lambda(x, y)$, where $\lambda(x, y)$ is a given function. If $w_x^2+w_y^2=\lambda(x, y)$, a particular solution is obtained by making $z+iw$ a function of $x+iy$. Some solutions which are not analytic over the whole of the basic region in the xy-plane are next considered, a simple type being obtained by using part of a surface $z=$const. and part of a surface $w=$const.

A more general way of solving $z_x^2+z_y^2=w_x^2+w_y^2$ is to put $u=z+w$, $v=z-w$ and to seek orthogonal nets of curves $u=$const., $v=$const. In polar coordinates the corresponding equation is $r^2z_r^2+z_\theta^2=r^2w_r^2+w_\theta^2$ and it is noted that, in the case of a helicoidal function $w=a\theta+f(r)$, the equation for z has a complete integral of helicoidal type

$$z=b\theta+g(r),\quad g(r)=\int(f'^2(r)+(a^2-b^2)/r^2)^{\frac{1}{2}}dr.$$

In the case when $\lambda(x, y)=e^{2F(x, y)}$, the substitution $z_x=e^F\cos\phi$, $z_y=e^F\sin\phi$ leads to the partial differential equation

$$\cos\phi\cdot\phi_x+\sin\phi\cdot\phi_y=F_y\cos\phi-F_x\sin\phi,$$

whose auxiliary equations

$$dx/\cos\phi = dy/\sin\phi = d\phi/[F_y\cos\phi - F_x\sin\phi]$$

lead to the ordinary differential equation

$$d^2y/dx^2 = [1+(dy/dx)^2][F_y-(dy/dx)F_x],$$

which is equivalent to one found by Darboux in a discussion of the analogy between geodesic lines and movements in a plane. A brief discussion of the equation is given and it is noted that there is a solvable case in which $E+iF=H(x+iy)$. Another integrable case is that in which $\lambda(x, y)=\mu(x)+\nu(y)$.

H. Bateman (Pasadena, Calif.).

Ledinegg, E. Über die allgemeinste Lösung der Maxwellschen Gleichungen in abgeschlossenen zylindrischen Räumen. Ann. Physik (5) **41**, 537–566 (1942). [MF 9236]

It is shown that Bromwich's method of solving Maxwell's equations for periodic processes by means of scalar potentials can provide the most general solution under prescribed boundary conditions in a cylindrical space with metallic walls, the proviso that the boundary is a member of a triply orthogonal system of surfaces being certainly fulfilled in the case of a cylindrical boundary whose cross-section is more or less arbitrary. For simply connected regions the field seems to be composed of one of electric type in which the magnetic field vector has no component in one direction and a field of magnetic type in which the electric field vector has no component in this direction. For a doubly connected region there is an additional component of Lecher type in which both the electric and magnetic field vectors have no component in this one direction. For the simply connected region the analysis is based largely on the theory of orthogonal functions. The discussion of the fields of Lecher type is based on the theory of conjugate functions.

H. Bateman (Pasadena, Calif.).

Dressel, F. G. and Elliott, E. R. A class of solutions for the heat equation and associated boundary value problems. Amer. J. Math. **65**, 408–422 (1943). [MF 8699]

Let D be the region of an x-y plane bounded by $y=h$, $y=g$ $(h<g)$ and $x=\chi_1(y)$, $x=\chi_2(y)$ $(\chi_1<\chi_2)$. In part I of their paper the authors deal with the problem of finding in D a solution $u(x, y)$ of the linear heat equation $u_{xx}=u_y$ satisfying the following boundary conditions

$$\lim_{\delta\to 0}\int_{x_1}^{x_2} u(t, 1/(h+\delta))\,dt = B(x_2)-B(x_1), \qquad \delta>0,$$

$$(*)\quad \lim_{\lambda_i\to 0}\int_{y_1}^{y_2} u_x(\chi_i(t)+\lambda_i, t)\,dt = G_i(y_2)-G(y_1), \qquad (-1)^{i+1}\lambda_i>0;\quad i=1, 2,$$

where B and G are given functions of limited variation. Under certain additional conditions concerning the possible discontinuities of these functions the authors prove the existence of a solution of this problem which is unique in a certain class of functions; the functions χ_i, $i=1, 2$, are supposed to have continuous second derivatives. The solution is given in terms of Young-Stieltjes integrals. [In a former paper [Amer. J. Math. **55**, 641–653 (1933)] the first named of the two authors treated a similar problem in which, however, the integrals in (*) contained the function u instead of the derivative u_x.]

In part II the authors prove the existence and, in a certain class, uniqueness of the solution of the classical boundary value of the first kind in the domain D for the heat equation under "parabolic approach" to the boundary if (i) χ_1, χ_2 have continuous first derivatives and (ii) the assigned boundary values are of limited variation and satisfy certain additional conditions concerning possible discontinuities. Here the term "parabolic approach" of a point P of D with coordinates x, y to a boundary point M means: if $M=(x_0, h)$, then $\lim_{P\to M}(x-x_0)/(y-y)^{\frac{1}{2}}=0$; if, however, $M=(\chi_i(y_0), y)$, $i=1, 2$, let E be the set of points of D through which P approaches M, and E_1 the subset of those points of E whose ordinates are greater than y_0. The approach of P to M is then called parabolic if either M is not a limit point of E_1, or if

$$\lim_{P\to M,\ P\subset E_1} |x-\chi_i(y_0)|/(y-y_0)^{\frac{1}{2}}=\infty.$$

The solution is expressed in terms of Riemann-Stieltjes integrals. *E. Rothe* (Ann Arbor, Mich.).

Lowan, Arnold N. A steady state problem in heat conduction. Philos. Mag. (7) **34**, 502–504 (1943). [MF 8882]

Lowan, Arnold N. A steady state problem in heat conduction. Revista Ci., Lima **44**, 499–504 (1942). [MF 8751]

The two papers are identical. A formula for steady temperatures in a semi-infinite hollow cylinder is formally obtained by the elementary method of separation of variables. *R. V. Churchill* (Ann Arbor, Mich.).

Lyubov, B. J. and Finkelstein, B. N. Designing of unstationary temperature fields in bodies of the simplest shape. I. Akad. Nauk SSSR. Zhurnal Eksper. Teoret. Fiz. **13**, 35–41 (1943). (Russian) [MF 9025]

Using the operational method, in particular the Bromwich integral, the authors give an explicit formula for the solution of $\nabla^2 T=a^{-1}\partial T/\partial t$ in a rectangle $0<x<2A$, $-B<y<B$ with $T(x, y, 0)=0$, $T(x, B, t)=\phi_1(x)t$, $T(x, -B, t)=\phi_2(x)t$, $T(0, y, t)=\psi_1(y)t$, $T(2A, y, t)=\psi_2(y)t$. *S. Bergman.*

Sutton, W. G. L. On the equation of diffusion in a turbulent medium. Proc. Roy. Soc. London. Ser. A. **182**, 48–75 (1943). [MF 9060]

The equation of diffusion

$$u\partial\bar{\chi}/\partial\bar{x} = (\partial/\partial\bar{z})(A_z\partial\bar{\chi}/\partial\bar{z})$$

is studied in the form

$$(1)\qquad \partial^2\chi/\partial z^2+[(1-2p)/z]\partial\chi/\partial z=\partial\chi/\partial x,$$

to which it was reduced with the aid of some assumptions equivalent to that of a power law $u=u_1(\bar{z}/z_1)^m$ for the mean velocity u. Sets of boundary conditions suitable for problems in evaporation are considered, one such set being $\lim_{x\to+0}\chi=0$ $(z>0)$, $\lim_{x\to+0}\chi=\chi_s$ $(0<x\leqq 1)$, $\lim_{x\to+0}\{z^{1-2p}\partial\chi/\partial z\}=0$ $(x>1)$. In this case initially dry air is supposed to travel over a saturated strip $0<x<1$ followed by dry ground $(x>1)$ impermeable to vapor. A Green's formula and a basic solution involving Bessel functions are found and it is shown that the basic solution can be deduced from the simple product solution by means of a transformation which is a simple generalization of Appell's transformation for the equation for the conduction of heat. A second transformation which is found to be useful is that, if χ satisfies (1), then the function $\chi'=z^{1-2p}\partial\chi/\partial z$ satisfies an associated equation with $1-p$ in place of p. When $0<p<1$, as is assumed in these investigations, the relation between the

two equations is a reciprocal one. The value $p=1/9$ corresponds to the case $m=1/7$ which was found so useful many years ago in the theory of turbulence.

Integrals are formed to satisfy initial conditions and the behavior of the function $F(z, x)$ represented by an integral involving the basic solution is investigated as $z\to\infty$ and as $z\to 0$ and as $x\to\infty$. Integrals are formed also to satisfy given boundary conditions on $z=0$ and the analysis is essentially that of integral equations of Abel's type and of the types considered by Poisson and Volterra. With the aid of a series of incomplete gamma functions an extension is found for the role played by the error function in the solution of problems of the conduction of heat. Integrals applicable to other regions than those previously considered are formed and some existence theorems are given.

The paper closes with an estimate of the magnitude of the correction required when the range of validity of the power law for fluid velocity u is restricted. Use is made of two integral equations and of the asymptotic expansions for the Bessel functions that are involved. Only the leading term in the correction is found and as this is 0.0017 it is thought that the full correction may not exceed 1% or 2% of the uncorrected solution. The nonhomogeneous equation is considered in connection with the two types of boundary value problem and some remarks are made on the differential equation for conditions which are not steady.

H. Bateman (Pasadena, Calif.).

Integral Equations

Fenyö, Stefan. Über die "Polynom-Kerne" der linearen Integralgleichungen. Math. Z. **48**, 772–780 (1943). [MF 9013]

A result obtained by T. Sato [Proc. Phys.-Math. Soc. Japan (3) **23**, 4–7 (1941); these Rev. **3**, 47] suggests the main theorem of this paper, namely, if

$$Q(x, y)=K_n(x, y)+a_1K_{n-1}(x, y)+\cdots+a_{n-1}K(x, y),$$

where $K_m(x, y)$ is the mth iterated kernel of $K(x, y)$ on $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, then a necessary and sufficient condition that λ be a characteristic value for $Q(x, y)$ is that one of the roots of

$$r^n+a_1r^{n-1}+\cdots+a_{n-1}r-1/\lambda=0$$

be a characteristic value for $K(x, y)$. The method of proof leads easily to a generalization concerning kernels Q which might be defined symbolically by the factored form

$$I-Q=(I-\lambda_1K^{(1)})(I-\lambda_2K^{(2)})\cdots(I-\lambda_nK^{(n)}),$$

where I is the identity transformation and the kernels are commutative. This includes as a special case kernels of the form

$$Q=K^{(1)}+K^{(2)}+\cdots+K^{(n)},$$

in which the kernels are mutually orthogonal and $Q=K_n$.

T. H. Hildebrandt (Ann Arbor, Mich.).

Eichler, M. Konstruktion lösender Kerne für singuläre Integralgleichungen erster Art, insbesondere bei Differenzkern. Math. Z. **48**, 503–526 (1942). [MF 8725]

The author considers the equation

$$(1)\qquad \pi f(x)=\int_{-1}^{1}L(x, \xi)\varphi(\xi)d\xi,$$

where

$$L(x, \xi)=(x-\xi)^{-1}+\sum_0^n c_j(x)q_j(x, \xi),$$

$$q_j=\frac{1}{j!}x^j\gamma_0(\xi)+\frac{1}{(j-1)!}x^{j-1}\gamma_1(\xi)+\cdots+\gamma_j(\xi)+\sigma_j(x-\xi),$$

$\sigma_0(x)=\ln|x|$, $\sigma_j(x)=\int_0^x\sigma_{j-1}(x)dx$. Integrals are in the sense of Cauchy principal values; all functions involved are continuously differentiable. The problem is of interest for certain physical applications and for the equation $\Delta u+u=0$. Functions $\Lambda(x, \xi)$, $N(\xi)$ are constructed so that

$$\varphi(\xi)=\pi^{-1}\int_{-1}^{1}\Lambda(\xi, t)f(t)dt$$

is a solution of (1), while $0=\int_{-1}^{1}LNd\xi$; $\Lambda(\xi, t)+N(\xi)\varphi_0(t)$, where $\varphi_0(t)$ is arbitrary, is the resolvent kernel of (1). Replacing $c_\nu(x)$, $\gamma_\nu(x)$ by $\lambda^{\nu+1}c_\nu(x)$, $\lambda^{-\nu-1}\gamma_\nu(\xi)$ and letting $\lambda\to 0$, by passing to the limit the author obtains a solution of the equation with the kernel

$$L^*(x, \xi)=(x-\xi)^{-1}+c_0(x)\gamma_0(\xi)+\cdots+c_n(x)\gamma_n(\xi).$$

The latter kernel is significant in view of the fact that every kernel $k(x, \xi)=(x-\xi)^{-1}+h(x, \xi)$, where h is piecewise continuous, can be uniformly approximated by kernels L^*.

W. J. Trjitzinsky (Urbana, Ill.).

Giraud, Georges. Équations de Fredholm dont le noyau est fonction holomorphe d'un paramètre; équations analogues où figurent des intégrales principales. C. R. Acad. Sci. Paris **212**, 36–38 (1941). [MF 9180]

In an earlier paper [Bull. Sci. Math. (2) **64**, 225–244 (1940); these Rev. **3**, 47] the author treated the homogeneous integral equation

$$u(x)=\int G(x, A; \lambda)u(A)dV_A,$$

where the integration is over an m dimensional region V and the kernel G is a holomorphic function of λ in a region D of the complex plane. The author states that his former representation of the resolvent of G can be simplified, and then writes down the sufficient conditions on $F(x)$ for the integral equation

$$u(x)=F(x)+\int G(x, A; \lambda)u(A)dv_A$$

to have a solution.

F. G. Dressel (Durham, N. C.).

Fock, V. Sur certaines équations intégrales de physique mathématique. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 133–136 (1942). [MF 8007]

Consider

$$(1)\qquad f(x)=g(x)+\int_0^\infty k(|x-y|)f(y)dy.$$

Reissner [J. Math. Phys. Mass. Inst. Tech. **20**, 219–223 (1941); these Rev. **3**, 151] determined solutions of (1) and Wiener and Hopf treated the case $g=0$. The writer's purpose is to give conditions for uniqueness and he announces that details will appear elsewhere. The methods seem obviously close to those used by Wiener and Hopf (though the writer works with L_1 functions of bounded variation rather than in L_2). Thus, for instance, the results hinge on the zeros of $1-K(w)$ in a certain strip, where $K(w)=Tk(x)$ and T

indicates a Fourier transform. Roughly speaking, if none of these zeros are real there is a unique solution satisfying $L_{x\to\infty}f(x)=0$, while, if $2n$ real zeros occur, $g(x)$ must be orthogonal to n determinate functions. *D. G. Bourgin.*

Kneschke, A. Wechselbeziehungen zwischen Differentialgleichungen und Integralgleichungen. II. Das allgemeine Integrationsproblem. Deutsche Math. 6, 543–553 (1942). [MF 8603]

The author extends the results of a previous paper [Deutsche Math. 5, 384–393 (1941); these Rev. 2, 311] to show that a certain type of integral equation of the second kind is equivalent to the boundary problem consisting of an nth order ordinary linear differential equation

$$f^{(n)}(\xi)+a_{n-1}(\xi)f^{(n-1)}(\xi)+\cdots+a_0(\xi)f(\xi)=g(\xi)$$

with prescribed initial values at the given points $x_1, \cdots, x_l$ ($l\leqq n$) of the n quantities $f(x_\mu)$, $f'(x_\mu)$, $\cdots$, $f^{(k_\mu-1)}(x_\mu)$, where $1\leqq k_\mu\leqq n$ ($\mu=1, \cdots, l$) and $k_1+\cdots+k_l=n$.

W. T. Reid (Chicago, Ill.).

Special Functional Equations

Bateman, H. Some simple differential difference equations and the related functions. Bull. Amer. Math. Soc. 49, 494–512 (1943). [MF 8765]

This paper is a survey of the various methods of solution of some ordinary and partial differential difference equations. The introduction considers briefly the following methods of solution: (a) separation of variables, (b) generating functions, (c) definite integrals, (d) operational methods, (e) power series, (f) successive approximations. In the second section the author indicates some simple applications of differential difference equations to analysis and the physical sciences. Save for one section which discusses the derivation of asymptotic forms, the remainder of the paper is concerned to a good extent with the study of the differential-difference equations in the theory of the compound pendulum, surges in springs and connected systems of springs, equations of Born and von Kármán [Phys. Z. 13, 297–309 (1912)] and elastic waves in a lattice.

A. E. Heins (Cambridge, Mass.).

Babini, José. Notes on arithmetic triangles. Univ. Nac. Tucumán. Revista A. 3, 302–310 (1942). (Spanish) [MF 9272]

Let $\phi(m, r)$ be defined for all nonnegative integers so as to satisfy the recurrence relation

$$\phi(m, r)=f_1\phi(m-1, r-1)+f_2\phi(m-1, r),$$

with f_1, f_2 functions of m and r. If further $\phi(0, 0)=1$ and $\phi(0, r)=0$ for $r>0$, the author speaks of the values of ϕ as forming an arithmetic triangle. [However, to complete the determination one must also specify $\phi(m, 0)$. From the context it appears that the recurrence relation is meant to be used with $\phi=0$ for negative r; thus $\phi(1, 0)=f_2(1, 0)$, etc.] After giving an explicit though somewhat complicated formula for ϕ, the author discusses a variety of special cases. Perhaps the most interesting of these is

$$\phi(m, r)=\Delta^r x(x-k)\cdots(x-mk+k)/r!h^r,$$

with $\Delta x=h$; this includes several well-known special results. The paper concludes with several evaluations of "generating" functions of the form

$$\sum_{r=0}^{m} a_r\phi(m, r)x(x-k)\cdots(x-mk+k).$$

There are a number of troublesome typographical errors.

I. Kaplansky (Cambridge, Mass.).

Cameron, R. H. and Martin, W. T. Infinite linear difference equations with arbitrary real spans and first degree coefficients. Trans. Amer. Math. Soc. 54, 1–22 (1943). [MF 8707]

The paper treats the equation

$$\sum_{n=1}^{\infty}(a_nz+b_n)f(z-\lambda_n)=g(z),$$

in which the λ_n are arbitrary real numbers and the coefficients a_n, b_n are complex and such that the series

$$\sum_{n=1}^{\infty}|a_ne^{\lambda_nw}|, \quad \sum_{n=1}^{\infty}(a_n\lambda_n+b_n)e^{\lambda_nw}$$

converge absolutely to suitable functions $A(w)$ and $B(w)$, respectively, in some strip $\alpha<\Re(w)<\beta$. The function $g(z)$ is taken to be analytic in some strip $a<\Im(z)<b$. The method is based upon an expression of the equation through the Stieltjes integral, and applications of the Fourier-Stieltjes transform. Theorems are stated separately for the case $a_n=0$, all n; for the case $a_n=0$, for all but one n, and for the general case. Their character is typified by that for the first of these cases which is as follows: If for some $\epsilon>0$ and every α', β', a', b' such that

$$(1)\qquad \alpha<\alpha'<\alpha+\epsilon, \quad \beta-\epsilon<\beta'<\beta, \quad a<a'<b'<b,$$

$$(2)\qquad \left.\begin{array}{ll} |g(x+iy)|e^{\beta'x}<K_{\alpha',\beta',a',b'} & \text{for } x<0 \\ |g(x+iy)|e^{\alpha'x}<K_{\alpha',\beta',a',b'} & \text{for } x>0 \end{array}\right\}, \quad a'<y<b',$$

the equation has a unique solution $f(z)$ that is analytic in $a<\Im(z)<b$ and satisfies relations obtainable from (2) by replacing g and K by f and K'. Moreover

$$f(z)=\sum_{n=1}^{\infty}b_n'g(z-\lambda_n'),$$

where

$$\frac{1}{B(w)}=\sum_{n=1}^{\infty}b_n'e^{\lambda_n'w}, \qquad \alpha<\Re(w)<\beta.$$

R. E. Langer (Madison, Wis.).

Le Caine, Jeanne. The linear q-difference equation of the second order. Amer. J. Math. 65, 585–600 (1943). [MF 9387]

This paper deals with the matrix q difference equation

$$(1)\qquad Y(qx)=(Ax+B)Y(x)$$

of the second order. The matrices A and B are constant and noncommutative, and $|q|>1$. On the basis of certain "characteristic constants" defined by it, the equation is classified as to type, and its reduction to a specified normal form through a linear transformation upon $Y(x)$ is described. A listing of the forms of fundamental solutions for equations of the several types is given. There is a discussion of the conditions under which two equations of the form (1) are transformable into each other by a rational transformation, and it is shown that for equations which are

related by a rational transformation there exist rational relations between their fundamental solutions and conversely. The Riemann problem is considered, and so also is the representation of a fundamental solution as an infinite matrix product. *R. E. Langer* (Madison, Wis.).

Anghelutza, Th. Sur une équation fonctionnelle. Bull. Sci. École Polytéch. Timişoara 11, 3 pp. (1943). [MF 9428]

The author shows that all real, continuous solutions of the system of equations

$$\varphi(x+y)=\varphi(x)\varphi(y)-\psi(x)\psi(y),$$
$$\psi(x+y)=\psi(x)\varphi(y)+\varphi(x)\psi(y)$$

(aside from the trivial solution $\varphi\equiv 0$, $\psi\equiv 0$) are contained in the form

$$\varphi(x)=b^x\cos ax,\quad \psi(x)=b^x\sin ax$$

($b>0$). The proof is based on a reduction of the system of equations to the single functional equation

$$f(x+y)+f(x-y)=2f(x)f(y).$$

F. John (Aberdeen Proving Ground, Md.).

Anghelutza, Th. Sur deux systèmes d'équations fonctionnelles. Mathematica, Timişoara 19, 19–22 (1943). [MF 9429]

The functional equation considered is

$$\varphi(x+y)-\varphi(x)-\varphi(y)=f(x,y).$$

F. John (Aberdeen Proving Ground, Md.).

GEOMETRY

Hall, Marshall. Projective planes. Trans. Amer. Math. Soc. 54, 229–277 (1943). [MF 9104]

A partial projective plane is a system of lines and points such that there is at most one line through two points and at most one point on two lines. If "at most one" is replaced by "exactly one" and the condition is added that there are four points no three of which are on a line, the plane is called complete. By adjoining to a partial plane π intersections of only two old lines and lines through only two old points a free extension of π is obtained. The free extension of π to a complete plane is unique. Equivalence of and invariants under free extensions are discussed. A plane is free and of type π^n if it is the free extension to a complete plane of $n-2$ points on a line and two points not on it. Relations between the various π^n are established. The paper then turns to the introduction of coordinates in non-Desarguesian planes. It is shown that coordinates depending on a fundamental quadrilateral and a ternary operation $x\odot m\odot b$ satisfying certain ring conditions can be introduced such that all lines have linear equations. Conversely, every ring satisfying these conditions leads to a projective plane. By means of these coordinates the configuration theorems of Hessenberg, Moufang and others are obtained and improved, and some new results are added. For the details the reader must be referred to the paper. *H. Busemann* (Chicago, Ill.).

Ancochea, G. Le théorème de von Staudt en géométrie projective quaternionienne. J. Reine Angew. Math. 184, 193–198 (1942). [MF 9030]

After deriving a theorem of von Staudt type when the fundamental field of a projective geometry is arbitrary (commutative or not) with characteristic not equal to 2, the author is concerned with such theorems for general quaternionic fields and finally with a specialization which includes ordinary real quaternions. Defining a projectivity between two first order forms as a biuniform correspondence preserving harmonic division, it is shown that for any field K a projectivity between two point ranges of the same line in which the three reference points are double points of the projectivity defines a semi-automorphism of the field (that is, a transformation $x'=(x)'$ such that $(x+y)'=x'+y'$ and $(xy)'+(yx)'=x'y'+y'x'$). Whether every semi-automorphism of K can be so obtained is, for general fields K, an open question. If K is general quaternionic, however, every semi-automorphism serves to define a projectivity between two point ranges of the same line in which the three reference points are double points. If, further, K is the field of real quaternions, every projectivity of von Staudt can be obtained by means of projections and sections, together with the operation of passing from a quaternion to its conjugate. *L. M. Blumenthal* (Columbia, Mo.).

Lee, H. C. The plane factorisation of pseudo-Euclidean rotations. Acad. Sinica Science Record 1, 45–49 (1942). [MF 8833]

The paper is devoted to proving the factorability of a rotation in Minkowskian geometry into a product of plane rotations, a known result. *G. de B. Robinson.*

Anghelută, Th. Une identité entre nombres complexes et un théorème de géométrie élémentaire. Bull. Sci. École Polytéch. Timişoara 10, 4 pp. (1942). [MF 9424]

This paper consists of two parts. First there is a short and direct proof, using complex numbers, of the theorem: given a regular polygon of n sides $A_1A_2\cdots A_n$ and a point I in the plane, then a polygon exists with sides $IA_1, IA_2, \cdots, IA_n$. The theorem was proved for $n=3$, $n=4$ by Pompeiu [Bull. Math. Phys. Éc. Polytéch. Bucarest 6, 6–7 (1936)]. A simple geometrical construction of the polygon is implicitly contained in the result. Secondly there is a proof that a certain inverse of the problem discussed in the first part, which has been established for $n=3$, cannot be extended to $n=4$ (given a triangle, then there exists an equilateral triangle $A_1A_2A_3$ and a point I in the plane such that IA_1, IA_2, IA_3 equal the three sides of the given triangle). *A. J. Kempner* (Boulder, Colo.).

M'Bride, James A. "The equal internal bisectors theorem, 1840–1940. . . . Many solutions or none?" A centenary account. Edinburgh Math. Notes no. 33, 1–13 (1943). [MF 9335]

This paper contains (i) a short history of the geometrical theorem proposed in 1840 by Professor Lehmus of Berlin to Jacob Steiner: "if BJY, CJZ are equal bisectors of the base angles of a triangle ABC, then AB equals AC," (ii) a selection of some half-dozen solutions from the 50 or 60 that have been given, (iii) some discussion of the logical points raised and (iv) a list of references to the extensive literature of the subject. *Author's summary.*

Nehring, O. Zyklishe Projektionen im Dreieck. J. Reine Angew. Math. 184, 129–137 (1942). [MF 9028]

Let A_1, B_1, C_1 be three points taken on the sides BC, CA, AB of a triangle ABC, and let 1 be an arbitrary point on

BC. Carry out the following projections $1B_1 \times BA = 2$, $2A_1 \times AC = 3$, $3C_1 \times CB = 4, \cdots$. The author shows that the operation closes with the second cycle, that is, $7 = 1$, if the Ceva relation $AA_1 \times BB_1 \times CC_1 = P$ is satisfied. Carry out analogous constructions with the points $1'$, $1''$ taken on the sides BC, CA and obtain the points $2'$, $3'$, $\cdots$, $7'$; $2''$, $3''$, $\cdots$, $7''$. If the points 1, $1'$, $1''$ are so chosen that $A1 \times B1' \times C1'' = Q$ the following propositions are shown to be true: $3B \times 3'C \times 3''A = R$, $5C \times 5'A \times 5''B = S$, the triads of points $(4, 4', 4'')$, $(6, 6', 6'')$ are collinear, etc. Analogous results are obtained when the points A_1, B_1, C_1 are collinear. In connection with this case the points of intersection of the sides of ABC with a conic are considered. Again, the author takes the point 1 on the line CC_1, carries out the constructions $1A_1 \times BB_1 = 2$, $2C_1 \times AA_1 = 3$, $3B_1 \times CC_1 = 4$, etc. and shows that $7 = 1$ if A_1, B_1, C_1 satisfy either the Ceva or the Menelaus relation. References are given to published works in which analogous constructions are applied to a quadrilateral, a hexagon, and a decagon. *N. A. Court.*

Santaló, Luis A. Some inequalities between the elements of a triangle. Math. Notae **3**, 65–73 (1943). (Spanish) [MF 9248]

Grossman, H. D. The Morley triangle: a new geometric proof. Amer. Math. Monthly **50**, 552 (1943). [MF 9533]

Thébault, V. Quadrangle bordé de triangles isoscèles semblables. Nat. Math. Mag. **18**, 7–13 (1943). [MF 9357]

Court, N. A. On the Cevians of a triangle. Nat. Math. Mag. **18**, 3–6 (1943). [MF 9356]

Court, N. A. A porism on eleven spheres. Math. Student **10**, 115–118 (1942). [MF 8741]

Soler, Pedro Rossell. Conoid with a spherical nucleus. Revista Union Mat. Argentina **9**, 49–54 (1943). (Spanish) [MF 9263]

Another solution of a problem previously discussed by Gaspar and Rey Pastor [same Revista **8**, 126–130, 131–135 (1942); these Rev. **4**, 166]. *P. Scherk.*

Flexner, W. W. Azimuth line of position. Amer. Math. Monthly **50**, 475–484 (1943). [MF 9258]

Ruiz, Jose Ignacio. Note on the precision of the construction of a point by the method of inverse trisection. Revista Acad. Colombiana Ci. Exact. Fis. Nat. **5**, 217–225 (1942). (Spanish) [MF 9096]

Concerning Pothenot's method in surveying.

Graf, Ulrich. Über die Verzerrungen beim plastischen Film. Jber. Deutsch. Math. Verein. **52**, 83–95 (1942). [MF 8537]

Stereoscopic slides (or films) may be projected on a screen. An observer viewing the screen through a suitable device (for instance, colored glasses or polaroid glasses, depending upon the construction of the projector) sees a three dimensional image. However, this image is correct only when the observer is at a definite position; for every other position of the observer the image is distorted. If the position of the observer as well as the length of the base (that is, the distance between the eye points) are changed, this distortion is a projective transformation. If the position of the observer is changed while the base is unchanged, the distortion is an affine transformation. The author shows that the image viewed from any position may be constructed by embedding the space of the observer into a four dimensional space. A "normal image" is then constructed in the three-space which intersects the space of the observer at right angles along the plane of the screen. To every position of the observer corresponds a point in the four dimensional space. When the "normal image" is projected from this point into the space of the observer, the image as seen by the observer is obtained. The author introduces a measure of distortion and shows that the surfaces of equal distortion are ellipsoids of revolution. *E. Lukacs.*

Graf, Ulrich. Bewegte Anaglyphen auf gewölbten Flächen. Deutsche Math. **6**, 394–408 (1942). [MF 8614]

A stereoscopic projection gives a correct image only when viewed from the position for which it was designed. When viewed from another point the image appears to be distorted. It is the general practice to draw stereoscopic projections on a plane. The author draws the two stereoscopic projections on a cylindrical surface in two complementary colors. This drawing must be viewed through appropriately colored glasses in order to produce a spatial image. When either the observer or the screen move the spatial image is distorted and performs a motion. The author discusses the transformation of the spatial image produced by moving the cylindrical screen and discusses several special cases (drawing the two projections on a circular cylinder or on a cylinder with a sine curve as a base) which were constructed according to his specifications. *E. Lukacs.*

Krames, Josef. Ueber die bei der Hauptaufgabe der Luftphotogrammetrie auftretenden "gefährlichen" Flächen. Bildmessung und Luftbildwesen **17**, 1–18 (1942). [MF 9342]

The paper is based on two lectures of the author and elaborates upon his article "Zur Ermittlung eines Objektes aus zwei Perspektiven" [Monatsh. Math. Phys. **49**, 327–354 (1941); these Rev. **3**, 300]. *E. Lukacs.*

Algebraic Geometry

Huff, Gerald B. A sufficient condition that a C-characteristic be geometric. Proc. Nat. Acad. Sci. U. S. A. **29**, 198–200 (1943). [MF 8443]

Necessary conditions that a set of integers $\{x_0, x_1, \cdots, x_n\} = \{x\}$, $x_1 \leqq x_2 \leqq \cdots \leqq x_n$, be the characteristic of a homoloidal net of plane curves are that they satisfy (1) the equations $x_1^3 + x_2^3 + \cdots + x_n^3 - x_0^3 = -1$, $x_1 + \cdots + x_n - 3x_0 = -3$, and (2) the inequalities which express the fact that no curve of order k can meet the net in more than kx_0 points. The author shows that condition (1) and a weaker form of (2) are sufficient. This weaker condition is (3) that the x's satisfy the finite set of inequalities $p_0x_0 \geqq p_1x_1 + \cdots + p_nx_n$, where $\{p\}$ runs over the characteristics of all homoloidal nets for which $p_0 < x_0$. The proof is based on the fact that any solution of (1) is transformed by A_{123}: $x_i' = x_i + x_0 - x_1 - x_2 - x_3$ into a solution of (1). Condition (3) is used to show that a repetition of this process reduces $\{x\}$ to $\{1; 0, \cdots, 0\}$. The homoloidal net of characteristic $\{x\}$ is then obtained from the product of the quadratic transformations associated with the A's. *R. J. Walker* (Aberdeen, Md.).

Perron, Oskar. Über die Bedingungen, dass eine binäre Form n-ten Grades eine n-te Potenz ist, und über die rationale Kurve n-ter Ordnung im R_n. Math. Ann. **118**, 305–309 (1942). [MF 8931]

A rational curve C_n of order n in projective R_n has the parametrization $x_i=s^i t^{n-i}$ and so is the (point-set) intersection of the n quadrics

$$\begin{vmatrix} x_{i-1} & x_i \\ x_i & x_{i+1} \end{vmatrix}=0, \qquad i=1, \cdots, n-1,$$

$$\begin{vmatrix} x_{n-1} & x_0 \\ x_n & x_1 \end{vmatrix}=0.$$

This property of C_n is strengthened in two ways: (1) C is the intersection of the $n-1$ surfaces

$$\begin{vmatrix} x_0 & x_1 & \cdots & x_k \\ x_1 & x_2 & \cdots & x_{k+1} \\ \cdot & \cdot & \cdot & \cdot \\ x_k & x_{k+1} & \cdots & x_{2k} \end{vmatrix}=0, \; k=1, \cdots, n-1,$$

where we have put $x_{n+1}=\cdots=x_{2n-2}=0$; (2) C is the intersection of $n-1$ quadrics if and only if n is a power of 2.

R. J. Walker (Aberdeen, Md.).

Perron, Oskar. Einige Bemerkungen über rationale Flächen. Math. Z. **48**, 467–496 (1942). [MF 8722]

By a "rational surface" in projective R_n the author means an $(n-1)$-dimensional variety V in R_n which is the rational (but not necessarily birational) transform of an R_{n-1}. Let $x_i=f_i(p_1, \cdots, p_n)$ $(i=0, \cdots, n)$ be the equations of the transformation, the f_i being homogeneous of degree m and having no common zero, and let P be the resultant of $x_0 f_j-x_j f_0$, $j=1, \cdots, n$, with respect to the p_j. Then $P=x_0^a F(x_0, \cdots, x_n)$, $a=(n-1)m^{n-1}$, where F is homogeneous of degree m^{n-1} and is a power of an irreducible polynomial $G(x_0, \cdots, x_n)$, $G=0$ being the equation of the variety V. If the f_i have common zeros F vanishes identically. For the case in which the f_i have a finite number of common zeros a method is given for determining a polynomial Q, of degree less than m^{n-1}, which behaves like F in relation to V. Fairly straightforward elimination theory is used for all the proofs. *R. J. Walker.*

Pa, Chen-Kuo. An analogue of Darboux pencil of quadrics. Acad. Sinica Science Record **1**, 65–69 (1942). [MF 8838]

The quadrics through the two asymptotic tangents at an ordinary point P of a nonruled irreducible surface S form an ∞^4 system. This system contains a pencil of quadrics F whose reguli belong to the osculating linear complexes of the asymptotic curves $u=c, v=c$ on S at P. To a quadric F_k of this pencil corresponds a quadric D_k of the Darboux pencil such that the product of the polarities with respect to F_k and D_k is an involution. The following constructions are made: the second directrix of Wilczynski by means of a given F_k and any D_k, the second projective normal of Fubini by an F_k, the second edge of Green if F_k is replaced by D_k in the preceding construction. *T. R. Hollcroft.*

Jung, Heinrich W. E. Rationale und halbrationale Doppelebenen. J. Reine Angew. Math. **184**, 199–237 (1942). [MF 8989]

Let (x, y, z) be a field of algebraic functions of 2 variables defined by the equation $z^2=W(x, y)$, where W is a polynomial in x, y. The problem discussed in this paper is that of determining (after applying suitable birational transformations) normal forms for W which insure that the field (x, y, z) is either rational or semi-rational. The first results of this type were discovered by Max Noether [S.-B. Phys.-Med. Soc. Erlangen **10**, 81–86 (1878)]. Later G. Castelnuovo and F. Enriques [Rend. Circ. Mat. Palermo **14**, 290–302 (1900)] reexamined and extended Noether's results. In order to recast the theory the author applies his method, which essentially consists in translating the geometric arguments of the Italian school into the analytic terminology of divisors. Thus some of the more intuitive arguments can be avoided. *O. F. G. Schilling.*

Muhly, H. T. Independent integral bases and a characterization of regular surfaces. Trans. Amer. Math. Soc. **54**, 340–360 (1943). [MF 9107]

Let Σ be a field of algebraic functions of r independent variables $x_1, x_2, \cdots, x_r$ over a ground field K of characteristic zero and let $\mathfrak{o}_0$ be the ring of integral functions of $x_1, x_2, \cdots, x_r$. Let $\mathfrak{o}_i$ denote the ring of integral functions of $1/x_i, x_1/x_i, x_2/x_i, \cdots, x_r/x_i$, $i=1, 2, \cdots, r$. It is proved that for any element ω in $\mathfrak{o}_0$ there exists an integer h, called the exponent of ω, such that $\omega/x_i^h \epsilon \mathfrak{o}_i$, $\omega/x_i^{h-1} \overline{\epsilon} \mathfrak{o}_i$, $i=1, 2, \cdots, r$. In terms of exponents of elements of $\mathfrak{o}_0$, the author generalizes the definition of normal integral bases of $\mathfrak{o}_0$ over $k[x_1, x_2, \cdots, x_r]$ which in the case $r=1$ is due to Dedekind-Weber. He calls such bases DW-bases. A DW-basis is at any rate an independent integral basis, and it is well known that, if $r>1$, independent integral bases do not always exist. The main result obtained by the author throws some light on the geometric significance of the existence of an independent integral base in the case of algebraic surface $(r=2)$. Quite generally, let $(\xi^*)=(\xi_0^*, \xi_1^*, \cdots, \xi_n^*)$ be the homogeneous coördinates of the general point of a normal variety V_r, and assume that the ξ^* are integrally dependent on $\xi_0^*, \xi_1^*, \cdots, \xi_r^*$. Then the nonhomogeneous coördinates $x_i=\xi_i^*/\xi_0^*$, $i=1, 2, \cdots, n$, are integrally dependent on $x_1, x_2, \cdots, x_r$. It is shown that the existence of an independent integral basis of $\mathfrak{o}^*=K[\xi^*]$ relative to the ring $K[\xi_0^*, \xi_1^*, \cdots, \xi_r^*]$ is equivalent to the existence of a DW-basis of $K[x]$ relative to the ring $K[x_1, x_2, \cdots, x_r]$. The main result is as follows: if $r=2$, then the existence of an independent integral basis for $\mathfrak{o}^*$ implies that the surface V_3 is regular (that is, $p_g=p_a$) and that essentially also the converse is true, provided that the surface is free from singularities. *O. Zariski* (Baltimore, Md.).

Jung, Heinrich W. E. Über ganze birationale Transformationen der Ebene. J. Reine Angew. Math. **184**, 161–174 (1942). [MF 9032]

Let (x, y), (x', y') be two rational function fields of 2 variables. A birational transformation between the fields is called integral if x, y are polynomials of x', y', and conversely. The author proves that each integral transformation is the product of transformations of the following types: (I) $x'=ax$, $y'=y/a$; (II) $x'=\pm y$, $y'=\pm x$; (III) $x'=x$, $y'=y+ax^\alpha$, α a positive integer. The proof is furnished by a study of the effect of the basic transformations on the degrees (with respect to the 2 variables) of the defining equations for a given integral transformation. As in the classical proofs of similar theorems it is necessary to count intersection numbers. *O. F. G. Schilling.*

Differential Geometry

DeCicco, John. Extensions of certain dynamical theorems of Halphen and Kasner. Bull. Amer. Math. Soc. **49**, 736–744 (1943). [MF 9315]

Certain theorems of Halphen and Kasner concerning the ∞^3 dynamical trajectories of a positional field of force in Euclidean 3-space are here considered in connection with the trajectories of generalized fields of force. These are fields in which the force depends upon both point and direction through the point. It is shown that, if each of the ∞^3 dynamical trajectories of a generalized field of force is a plane curve, the curves consist of ∞^2 families of ∞^1 generalized plane trajectories, each such family lying in a plane tangent to a fixed surface Σ. Various degenerate cases are described. In the corresponding theorem of Halphen for ordinary fields of force Σ degenerates into a fixed point. In certain generalized fields of force, the rest trajectory and line of force at any point have the same osculating plane. In these fields it is proved that the ratio of the curvature of the rest trajectory to that of the line of force is $(1-\lambda)/(3-\lambda)$, where λ depends upon the field. In the corresponding theorem of Kasner $\lambda=0$. *A. Fialkow.*

DeCicco, John. Geometry of dual-velocity systems. Univ. Nac. Tucumán. Revista A. **3**, 261–270 (1942). [MF 9269]

By a dual-velocity system the author means a family of ∞^2 plane curves which satisfy an equation of the form

$$d^2y/dx^2=a(x, y)\cdot dy/dx+b(x, y),$$

where x, y are the equilong coordinates of a straight line. It is proved that a family F of ∞^2 curves is a dual-velocity system if and only if the centers of the osculating circles of the ∞^1 curves of F tangent to any line l, constructed at the lineal elements of l, describe a straight line. The complete group of line transformations which convert every dual-velocity system into a similar system is the dual-isothermal group. Various special types of dual-velocity systems are considered in greater detail. *A. Fialkow.*

Kasner, Edward and DeCicco, John. An extensive class of transformations of isothermal families. Univ. Nac. Tucumán. Revista A. **3**, 271–282 (1942). [MF 9270]

It is known that the only point transformations of the plane which convert every isothermal family of curves into an isothermal family are the conformal transformations. In the present paper the authors determine all "field-element" transformations which preserve isothermal character. A field-element is a number quintuplet (x, y, θ, r, s), where (x, y) are the Cartesian coordinates of the point, θ is the inclination of the direction, $r=\theta_x$, $s=\theta_y$. In the complex plane there are seven distinct types of field-element transformations and nine distinct types of contact field-element transformations which carry every isothermal family into an isothermal family. The corresponding results for the real plane are also obtained. *A. Fialkow.*

Sauer, Robert. Streifenmodelle und Stangenmodelle zur Differentialgeometrie der Drehflächen, Schraubenflächen und Regelflächen. Math. Z. **48**, 455–466 (1942). [MF 8721]

Baier, Othmar. Zum Meusnierschen Satz. Math. Z. **49**, 148–150 (1943). [MF 9193]

Chang, Su-Cheng. Some theorems on ruled surfaces. Acad. Sinica Science Record **1**, 75–77 (1942). [MF 8841]

In 1927 L. Godeaux obtained a sequence of quadrics φ_i associated with a point of a surface. The first quadric φ_1 is the quadric of Lie. In 1935 B. Su defined a quadric φ_2 which Godeaux two years later showed to be the second of his sequence. In the present paper three theorems on ruled surfaces are proved and from them the following theorem 4: both sequences of quadrics associated with the two asymptotic ruled surfaces of an analytic nonruled surface at a generic point coincide with the sequence φ_i.

T. R. Hollcroft (Aurora, N. Y.).

Gheorghiu, Gh. Th. Sur les courbes de Tzitzéica. Mathematica, Timişoara **19**, 97–105 (1943). [MF 9423]

A Tzitzéica curve (T) is characterized by $\tau=\text{const.}$ d^3 (τ = torsion, d = polar distance to osculating plane). Using a centro-affine approach the present paper studies Tzitzéica curves (TQ) which lie on a centered quadric. Tzitzéica showed that among the asymptotic transforms of a given curve (T) there are ∞^3 curves (T). The author finds that among the asymptotic transforms of a curve (TQ) there are in general just two other curves (TQ) and all three lie on the same centered quadric. From this fact an iteration process emerges, generating from a given curve (TQ) a unique infinite sequence of curves (TQ) belonging to the same quadric. Some properties of these sequences are developed. Tzitzéica also showed that the ruled surface generated by the lines joining corresponding points of a curve (T) and one of its (T) asymptotic transforms has among its asymptotic lines only these two curves as (T) curves, or else all its asymptotic lines are (T) curves. In the first case it is a surface (S) with one flecnodal tangent at infinity. The author finds that for curves (TQ) the first case always holds, leading to a subclass of the surfaces (S) which he plans to study further. *J. L. Vanderslice.*

Silberstein, Ludwik. Infinitesimal bending of a surface. Philos. Mag. (7) **34**, 549–554 (1943). [MF 9135]

The author applies the equations of Killing to a study of the infinitesimal bending of surfaces in three dimensional Euclidean space. The components of the displacement vector for the axial symmetric bending of a surface of revolution are reduced to expressions requiring one quadrature for their evaluation. Further, the displacement vector for the general infinitesimal bending of the sphere is determined. By comparing these two results, the author concludes that the sphere cannot be bent into a surface of revolution (with or without discontinuity of the displacement vector). After eliminating this possibility of axial symmetric bending of a sphere, the author furnishes an example of nonaxial symmetric bending of the sphere for a discontinuous displacement vector. *N. Coburn* (Austin, Tex.).

Ward, L. E. Geodesics and plane arcs on an oblate spheroid. Amer. Math. Monthly **50**, 423–429 (1943). [MF 9091]

The author compares the length of a geodesic arc P_1P_2 on an oblate spheroid with the length of the elliptic arc P_1P_2 which is the intersection of the spheroid with the plane determined by P_1P_2 and the center of the spheroid. He finds that the difference of these two lengths is an infinitesimal of order four referred to the eccentricity of the spheroid. *S. B. Myers* (Ann Arbor, Mich.).

Choudhury, A. C. On a case of the cross ratio system of a 3-web. Bull. Calcutta Math. Soc. **34**, 177–181 (1942). [MF 8999]

A 3-web of curves consists of three one-parameter families of curves. A 3-web of curves determines a fourth one parameter family, in the sense that, the cross-ratio of the tangents to the four being given, the fourth is determined. Thus, if $u=\text{const.}$, $v=\text{const.}$, $du/dv=f(u,v)$ form a 3-web, then the curves of the family defined by $du/dv=\sigma f(u,v)$ form with the curves of the 3-web the cross ratio σ. The curves of the two parameter family of curves defined by

$$\frac{d^2u}{dv^2}=F_u\left(\frac{dy}{dv}\right)^2+F_v\frac{du}{dv},$$

wherein $F=\log f$, possess the cross-ratio property. This two parameter system of curves is called a cross-ratio system. It is shown that a necessary and sufficient condition that the cross-ratio system of a 3-web may contain a hexagonal web $u=\text{const.}$, $v=\text{const.}$, $u+v=\text{const.}$ is that $F_u=F_v=\varphi(u+v)$. Finally it is shown that, if the 3-web generating the cross-ratio system is also hexagonal, then the cross-ratio system is reducible to straight lines. Two distinct types of such systems are obtained. *V. G. Grove.*

Su, Buchin. On certain pairs of surfaces in ordinary space. Bull. Amer. Math. Soc. **49**, 722–729 (1943). [MF 9311]

On what surfaces S in ordinary space does there exist a two parameter family of curves (i) forming triangles all of angular excess kA ($A=$area, $k=$constant) and (ii) forming a map of the geodesics of some surface S_1? This is a generalization of a problem recently proposed and solved by J. Douglas [Trans. Amer. Math. Soc. **48**, 101–116 (1940); these Rev. **2**, 19]. There S_1 was a plane. The conditions for (i) obtained by Douglas are written in an orthogonal coordinate net on S. Using the corresponding (Tissot) orthogonal system on S_1 as parametric, condition (ii) takes analytic form as differential equations on E, G, E_1, G_1, the coefficients of the linear elements ds^2, ds_1^2 of S, S_1. In case $E/G=E_1/G_1$ it is found that $ds^2=(K_1/k)ds_1^2$ ($K_1=$curvature of S_1, k as above). When $k=0$ then $K_1=0$, a special case of Douglas' problem. In case $E/G\neq E_1/G_1$, the differential equations are integrated leading ($k\neq 0$) to explicit forms for ds^2, ds_1^2 involving one arbitrary function. After $k=0$ is given special treatment the problem stands completely solved.

J. L. Vanderslice (College Park, Md.).

Su, Buchin. Plane sections through a point of a nonholonomic surface. Amer. J. Math. **65**, 701–711 (1943). [MF 9396]

This is an extension to a nonholonomic V_3^2 ($dz=p(x,y,z)dx+q(x,y,z)dy$) in projective S_3 of a type of investigation already applied to surfaces in previous contributions of the author. The first main result is the theorem: if two planes through a point O of V_3^2 each pass through one of two arbitrary but fixed nonasymptotic tangents so that the osculating conics of the sections of V_3^2 by these planes lie on a quadric Q tangent to V_3^2 at O, then Q has second order contact at O and the correspondence between the two planes is (2, 2). In the theorem for the holonomic case [Amer. J. Math. **65**, 439–449 (1943); these Rev. **5**, 14] this correspondence is a perspectivity. In the present case each Q contains four additional conics osculating sections of V_3^2. Each pair of nonasymptotic tangents at O determine through the (2, 2) correspondence a line and quadric cone, third and fourth order covariants. From these emerge a third order covariant quadric and line at O depending only on V_3^2. These configurations do not exist for surfaces. The paper concludes with derivation of several geometric properties of these lines and cones arising from selection of the original two tangents in an invariant way, for example, as Segre tangents. *J. L. Vanderslice* (College Park, Md.).

Su, Buchin. A generalization of the canonical quadric of Wilczynski in the projective theory of non-holonomic surfaces. Univ. Nac. Tucumán. Revista A. **3**, 351–362 (1942). [MF 9276]

For investigating the neighborhood of a point O of a nonholonomic V_3^2: $dz=p(x,y,z)dx+q(x,y,z)dy$, the author uses a projective reference system determined by the asymptotic directions there. Also used are Bortolotti's nonholonomic derivatives $\partial_1=\partial_x+p\partial_z$, $\partial_2=\partial_y+q\partial_z$ and asymptotic derivatives $\partial_u=\partial_1+B\partial_2$, $\partial_v=\partial_2+C\partial_1$ (B, C well-defined functions of x, y, z). Any curve C through O on V_3^2 is defined through its direction field $dv/du=\lambda(x,y,z)$. With this machinery the equations of the two asymptotic osculating quadrics defined at O by C are exhibited. These equations and those of the other quadrics arising in the investigation involve as coefficients only p, q, B, C, λ and their first and second ordinary or nonholonomic derivatives at O. When C is in turn the u, v asymptotic curve through O the two resulting quadrics generate a pencil in which the Bortolotti quadric is the one having second order contact with V_3^2 at O. When C is in turn tangent with inflection point to the asymptotic curves at O a characterization of the nonholonomic analogue of the Wilczynski quadric appears in exactly similar manner. Also based on the asymptotic osculating quadrics is a new characterization of the Darboux quadrics together with two distinct nonholonomic generalizations of the Fubini quadric.

J. L. Vanderslice (College Park, Md.).

Green, L. The axial quadrics of a surface. Duke Math. J. **10**, 557–564 (1943). [MF 8980]

A quadric Q is said to be axial if it has second order contact with a surface S at a point x and if the three osculating planes of the curve of intersection of Q and S are coaxial. There is a two parameter family of axial quadrics of S at x. Each line protruding from S at x is an axis of at least one axial quadric. If the condition that a quadric be axial is applied to the quadrics of Darboux, that quadric must be the quadric of Wilczynski, and the axis is a certain line of the first canonical pencil [L. Green, Amer. J. Math. **60**, 649–666 (1938)]. Applications are also made to the quadrics of Davis [W. M. Davis, Contributions to the theory of conjugate nets, Dissertation, Chicago, 1933] and to the asymptotic osculating quadrics of a curve on S.

V. G. Grove (East Lansing, Mich.).

Hsiung, Chuan-Chih. Projective differential geometry of a pair of plane curves. Duke Math. J. **10**, 539–546 (1943). [MF 8976]

Let C_1, C_2 be two analytic plane curves tangent to a line O_1O_2 at O_1 and O_2, respectively, O_1 and O_2 being ordinary points on their respective curves. A third vertex O_3 of a canonical triangle is determined as the intersection of the polar lines of the points O_1, O_2 with respect to the conics of two pencils of conics having third order contact with C_1 and C_2, respectively. Projective invariants I_1, I_2 are determined as cross-ratios of points associated with points and

lines of conics having third order contact with C_1, C_2 and nodal cubics having fifth order contact with these curves. Canonical power series expansions of C_1, C_2 are established. Finally it is shown that the limiting position of the lines O_1O_3 (or O_2O_3) as O_3 (or O_1) approaches O_1 (or O_2) along O_1O_2 is the covariant line $r_0^{(1)}$ of Bompiani [Accad. Naz. Lincei. Rend. (6) 3, 118–123 (1926)]. *V. G. Grove.*

Lagrange, René. Sur les invariants conformes d'une courbe. C. R. Acad. Sci. Paris 212, 1123–1126 (1941). [MF 9188]

A clever method of obtaining the conformal curvature and torsion of a curve in Euclidean three-space. At each point of the curve is a moving pentasphere (an analogue of the moving trihedral) and use is made of the fact that each conformal subgroup leaving a sphere invariant is the group of a Riemann space, so that the conformal invariants are obtained as the metric invariants of this Riemann space.

M. S. Knebelman (Pullman, Wash.).

Lagrange, René. Propriétés différentielles des courbes de l'espace conforme à n dimensions. C. R. Acad. Sci. Paris 213, 551–553 (1941). [MF 9222]

This note extends the methods and results obtained in the paper reviewed above to a curve in an n-dimensional Euclidean space. The author obtains the conformal Frenet equations in terms of the successive curvatures of the associated Riemann space. *M. S. Knebelman.*

Lee, H. C. On unitary geometry. Acad. Sinica Science Record 1, 49–54 (1942). [MF 8834]

The main purpose of the present note is to establish the result that, if a (generalized) unitary space is symmetric, its fundamental tensor is the second derivative of a semi-analytic function of the (complex) coordinates. The generating functions for flat unitary spaces and for unitary spaces of constant curvature are also determined.

Author's introduction.

Maass, Hans. Über eine Metrik im Siegelschen Halbraum. Math. Ann. 118, 312–318 (1942). [MF 8933]

Let $Z=X+iY$ be an n-rowed symmetric matrix with complex elements and let the imaginary part Y of Z be the matrix of a positive definite quadratic form. The space $\mathfrak{H}$ of all these Z is transformed into itself by all mappings

$$Z\to(AZ+B)(CZ+D)^{-1},$$

where A, B, C, D are n-rowed real matrices satisfying $AB'=BA'$, $CD'=DC'$, $AD'-BC'=E$. The author proves that the quadratic differential form $ds^2=\text{trace}\,(Y^{-1}dZ\,Y^{-1}d\bar{Z})$ is invariant under (1) and that the corresponding metric defines, for any two points in $\mathfrak{H}$, a uniquely determined geodesic arc; moreover, he obtains simple expressions for the geodesic distance and the equation of the geodesic lines.

C. L. Siegel (Princeton, N. J.).

Lotze, A. Nachtrag zu meinem Aufsatz: "Die elementaren Differentialoperationen in der Grassmannschen Vektoranalysis." Jber. Deutsch. Math. Verein. 52, 245–250 (1942). [MF 9056]

In an earlier article [same Jber. 50, 79–91 (1940); these Rev. 2, 254] the author derived various invariants by use of Grassmann's methods. The present paper continues some of these previous developments. In particular, the curl, divergence, and gradient are discussed. Finally, a few corrections of some results in former papers by the author are given. *N. Coburn* (Austin, Tex.).

Platrier, Charles. Composition des dérivées pour un tenseur euclidien d'un champ relatif. C. R. Acad. Sci. Paris 215, 456–457 (1942). [MF 9511]

The author states two theorems concerning the relations between the absolute and relative components of the first and second derivatives of a Euclidean tensor, of order n in three-dimensional space, with respect to a scalar variable t. The theorems are generalizations of the familiar kinematical relations between the absolute and relative components of the velocity and the acceleration of a point.

L. A. MacColl (New York, N. Y.).

* **Craig, Homer V. Vector and Tensor Analysis.** McGraw-Hill Book Co., Inc., New York, 1943. xiv+434 pp. $3.50.

"There might be a place for a text on vector and tensor analysis which (1) would be available to people who are not thoroughly prepared in advanced calculus and yet avoid the harmful use of differentials as small quantities, (2) would apply vector analysis to topics in advanced calculus and to tensor analysis, and use tensor notation in vector analysis and advanced calculus and (3) would emphasize invariance in functional form." This excerpt from the author's prefatory apology is a succinct summation of his book. He admits indebtedness for ideas to no less than forty-one mathematicians, confesses to one or two minor innovations and apologizes for the inclusion of extensors (a discipline initiated by him) because of a desire to stress invariantive procedures. As a matter of fact the book has many original features and the extensor idea is deserving of publicity for its own sake.

Part A [Mathematical Background, 108 pages] is essentially a brief treatise on advanced calculus with overtones of real variable theory. Noteworthy is the rigorous presentation (unusual for works with an eye on applications), the emphasis on the theorems of the mean as basic tools, the complete discussion of multiple integrals and the consistent avoidance of irrelevant material. There are special chapters on parameterized arcs, determinants and the alternating e-quantities and coordinate transformations. Tensors put in an initial appearance in connection with the chain rule of differentiation. Part B [Elementary Vector Analysis, 133 pages] starts with an informal but full discussion of vector analysis of three dimensions as a physico-geometric rather than as a logical discipline. Emphasis is placed on its invariantive aspects, its compactness and utility for simplifying proofs and statements of theorems. Axiomatic formalities enter with Euclidean n-space regarded as a linear vector space with a special quadratic metric. Orthogonal bases receive original treatment. A chapter on vector calculus includes besides the usual material a touch of the differential geometry of curves and surfaces. Here also is an introductory discussion of extensors as tensors under an extended point transformation. Another chapter on vector integral calculus succeeds in straightening out for the reader the endless variety of Green's theorems.

Part C [Tensors and Extensors, 68 pages] covers the traditional material of tensor algebra and calculus including covariant differentiation and Riemann tensor, but the methods are not all traditional. For example, extensors are discussed simultaneously (perhaps for the first time in a textbook) and the covariant derivative and fundamental affine connection emerge from them. This mode of presentation is certainly controversial, at least as a first approach

to the subject. The author considers his method less fortuitous than the usual one but for good measure outlines the latter. For the most part the tensor development is made on curves and surfaces of Euclidean vector space; consequently we find vector and tensor notation running hand in hand. Parallelism, geodesics, geodesic coordinates and conditions for flatness receive special attention. Throughout some of Schouten's notational ideas are used to good advantage. Part D [Applications, 107 pages] deals first with classical dynamics with the intention of later generalization to relativity theory. Vector methods are used throughout with tensors creeping in as usual to care for strained media. The theorem of the mean is used to lend a certain rigor to the derivation of the fundamental dynamical equations of continuous media. The final two chapters are on special and general relativity. Because of the extensive preparation a satisfactory treatment takes a mere 63 pages. Some elements in the presentation seem novel, for example, the introduction of a time scale.

Each section of the book has its own bibliography, principally of well-established works, while references in the body of the text are few. There is a generous number of problems. They are in the same typography as text proper probably because many important developments are relegated to them, for example, certain generalizations of Riemannian geometry. Some seem to have been inserted as master thesis material. An unusual conceit is the weighting of the problems by giving lengthy ones more than one number. As a final word perhaps the most noteworthy characteristics of the book are (1) the unusual rigor for a work on the physical borderline of mathematics and (2) the smooth coordination of a large number of modern viewpoints under the aegis of invariance.

J. L. Vanderslice (College Park, Md.).

MECHANICS

Dugas, René. Choc des corps et moindre action. C. R. Acad. Sci. Paris **214**, 899–900 (1942). [MF 9463]

The principle of least action was stated by Maupertuis in 1744 and applied by him to the direct collision of bodies either perfectly elastic or perfectly inelastic [Mém. Acad. Berlin 2, 267–294 (1746)]. The present paper extends the method of Maupertuis to direct and indirect collisions of partially elastic bodies. *G. E. Hay* (Ann Arbor, Mich).

Platrier, Charles. Sur un postulat de la statique et sur une extension de la mécanique newtonienne. C. R. Acad. Sci. Paris **214**, 973–975 (1942). [MF 9471]

The author considers a variation of Newtonian dynamics in which the equation $md^n\gamma/dt^n=d^nF/dt^n$, rather than the equation $m\gamma=F$, is taken as fundamental (m denotes the mass, γ the acceleration of the particle, F the applied force and n is a fixed integer). The differential equations of motion are given, for the case in which $n=1$, in a form which is similar to that of the ordinary Lagrangian equations. Formulae are given for the components of the vector $d\gamma/dt$ along the tangent, principal normal and binormal of the trajectory. A theorem similar to the Coriolis theorem is stated, relating to the absolute and relative components of $d\gamma/dt$. *L. A. MacColl* (East Lansing, Mich.).

Darevsky, V. M. The Gibbs phenomenon for dynamic effect. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **7**, 71–73 (1943). (Russian. English summary) [MF 9728]

Hamel, Georg. Über die ebene Bewegung eines unausdehnbaren Fadens. Math. Z. **48**, 27–47 (1942). [MF 8560]

The author investigates certain special motions of a heavy string in a vertical plane, with and without stiffness, also the general motion for the perfectly flexible case. The solutions are valid for small values of the time and are obtained by a successive approximation procedure. In the first special case, certain paradoxical conclusions are avoided by the assumption of a discontinuity in the curvature.

P. Franklin (Cambridge, Mass.).

Chazy, Jean. Sur une généralisation du pendule cycloidal d'Huygens. C. R. Acad. Sci. Paris **213**, 93–98 (1941). [MF 9155]

The author constructs unsymmetrical one dimensional force fields for which the period of a vibrating particle is independent of the amplitude, and vertical plane curves on which the motion under gravity is of this type.

P. Franklin (Cambridge, Mass.).

Magnus, K. Über die Anwendungen der allgemeinen Bewegungsgleichungen starrer Körper in bewegten Bezugssystemen. Z. Angew. Math. Mech. **22**, 336–356 (1942). [MF 9120]

For the study of the dynamical properties of the top and pendulum-like instruments one must frequently apply the general dynamical equations for the motion of a rigid body. The author applies the "motor calculus" to the studies of such problems and treats in particular (a) the physical pendulum and (b) a three body problem, namely, a force free top with inner and outer Cardan rings. It is remarked that the rigorous equations are usually too difficult to solve, and accordingly some approximations under which the equations of motion can be solved are discussed.

A. E. Heins (Cambridge, Mass.).

Timmel, E. Grenzen für die Genauigkeit der Lotbestimmung im Flugzeug durch Scheinlotmittelung. Luftfahrtforschung **20**, 171–174 (1943). [MF 9127]

The accuracy of determining the vertical is closely related to the accuracy of astronomical navigation. The instrument used to determine the vertical generally measures the direction of the apparent gravitational acceleration of the aircraft, and therefore the measurement contains an error due to the nonuniform velocity of flight. The author investigates the possible error in the direction of vertical due to two types of disturbances. The first type of disturbance is periodic in nature and is the result of oscillations of the compass, the automatic pilot, etc. The error due to this type of disturbance could be reduced by lengthening the measuring interval and taking the average. The second type of disturbance is not periodic and is the result of Coriolis acceleration, centrifugal acceleration due to curvature of the flight path or change in wind velocity.

H. S. Tsien.

Andress, W. R. A vector account of four-piece mechanisms. Math. Gaz. **27**, 149–154 (1943). [MF 9525]

Geiringer, Hilda. The geometric foundations of the mechanics of a rigid body. Amer. Math. Monthly **50**, 492–502 (1943). [MF 9261]

Expository article.

Sályi, St. Nachtrag zu meinem Aufsatz in Band XIII, S. 104, des Ingenieur-Archivs "Ein graphisches Verfahren zur Bestimmung der Eigenfrequenzen von Systemen mit mehreren Massen." Ing.-Arch. **13**, 330 (1943). [MF 9073]

The paper appeared in the Ing.-Arch. **13**, 104–109 (1942); these Rev. **5**, 17.

d'Adhémar, Robert. La stabilité du projectile tournant. La tenue. L'amortissement initial rapide. C. R. Acad. Sci. Paris **213**, 17–19 (1941). [MF 9152]

The term stability is used in ballistics in different and sometimes contradictory senses. The author proposes to limit the use of the word stability to its strictly technical meaning in mechanics, and to employ the term steadiness (that is, "holding to its path," French la tenue) for the case where the yaw remains small. The paper also points out misconceptions concerning damping. *W. E. Milne.*

Astronomy

Sémirot, Pierre. Chocs imaginaires dans le problème des trois corps. C. R. Acad. Sci. Paris **212**, 974–977 (1941). [MF 9178]

The solutions of the three body problem are considered from the point of view of complex variables, the coordinates being expanded as power series in a complex time variable t. Singularities, called imaginary collisions, can arise if one, two or three of the "distances" r_{ij} vanish for an imaginary value of t, while no two particles coincide for this t. These were first studied by Chazy [C. R. Acad. Sci. Paris **157**, 1398–1400 (1913)]. In the present paper the author studies such singularities from the point of view of Chazy's generalization of the Poincaré classification of the singular points of a second order system [see Chazy, Bull. Sci. Math. (2) **56**, 79–104 (1932)]. The roots of the characteristic equation are found in each case and the form of expansion of the solution in terms of t is given. *W. Kaplan.*

Belorizky, David. Chocs d'une nouvelle espèce dans le problème des trois corps. C. R. Acad. Sci. Paris **213**, 558–560 (1941). [MF 9176]

A mixture of real and imaginary collisions is considered [see the preceding review] in the sense that for a certain (complex) value of t two bodies actually coincide (that is, their complex coordinates are the same), while all three of the distance functions vanish. Expansions for the coordinates in terms of t in the neighborhood of the singularity are given, and the transcendental nature of the singularity is pointed out. *W. Kaplan* (Ann Arbor, Mich.).

Holtom, Carl. Permanent configurations in the n-body problem. Trans. Amer. Math. Soc. **54**, 520–543 (1943). [MF 9524]

The permanent configurations of the n-body problem are those solutions for which the ratios of all mutual distances remain constant. General properties and special cases have been studied by many authors [see A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton Mathematical Series, vol. 5, Princeton University Press, Princeton, N. J., 1941, pp. 273–306, pp. 430–431; these Rev. **3**, 215]. In the present paper it is proved that, subject only to certain inequalities on the sizes of the masses, there always exist planar permanent configurations. The method of proof consists in first establishing existence of a position (libration point) at which an nth infinitesimal mass can be added to $n-1$ finite masses assumed in a permanent configuration, and then in showing that this solution can be continued analytically for small positive values of the nth mass. Series expansions for the solutions in terms of the mass ratios are found and illustrations are given for $n=3$, 4 and 5. *W. Kaplan.*

***Banerji, A. C. Recent Advances in Galactic Dynamics.** Lucknow University Studies, no. 15. Allahabad Law Journal Press, Allahabad, India, 1942. iv+113 pp.

This monograph consists of three chapters dealing, respectively, with the theories relating to the polytropic configurations of gaseous matter, spiral nebulae and the origin of the solar system.

In the first chapter the principal topic of discussion centers around the modern investigations (Milne, von Zeipel, Chandrasekhar, Bhatnagar and others) on the equilibrium of rotating compressible masses. In the second chapter the various dynamical theories which have been advanced in recent years to account for the spiral structure of the extragalactic nebulae are reviewed, particular attention being paid to those investigations which have further developed Lindblad's original suggestion that the spiral structure may have arisen from the instability of the circular orbits at the "edges" of the systems. Chapter III critically discusses the existing theories concerning the origin of the solar system. The difficulties confronting the tidal theories in particular are exposed in some detail. Toward the end of this chapter the author refers to his own theory in which the sun is presumed to have originally been a Cepheid variable; however, the section dealing with this new theory is too brief to judge its merits. [The author's original paper on the "Cepheid-theory" of the origin of the solar system appears to have been published in the Proc. Nat. Inst. Sci. India **8**, 173–197 (1942); these Rev. **4**, 117, but this paper is inaccessible to the reviewer.]

S. Chandrasekhar (Williams Bay, Wis.).

Mayot, Marcel. Stabilité des figures d'équilibre d'un amas d'étoiles dont le centre décrit un cercle dans la voie lactée. C. R. Acad. Sci. Paris **212**, 45–47 (1941). [MF 9183]

The secular equilibrium of ellipsoidal homogeneous distributions of mass sharing in galactic rotation is considered along the lines familiar in the theory of the equilibrium of incompressible fluids. [However, it appears to the reviewer that treating stellar systems as "incompressible fluids" is highly misleading and dangerous.] *S. Chandrasekhar.*

Kothari, D. S. The source of energy in a white dwarf star. Science and Culture **6**, 309–310 (1940). [MF 9673]

Hydrodynamics, Aerodynamics

Goudet, Georges. Étude des ondes stationnaires ultrasonores dans les liquides. C. R. Acad. Sci. Paris **213**, 117–119 (1941). [MF 9157]

Ballabh, Ram. Steady uniplanar superposable fluid motions. J. Indian Math. Soc. (N.S.) **7**, 36–41 (1943). [MF 9379]

Let $(u', v', 0)$ be a steady irrotational motion and $(u, v, 0)$ a steady rotational motion with ξ, η, ζ as components of vorticity. Since u' and v' do not depend on z the conditions for the superposability of the first motion on the second are

$u'\zeta_x+v'\zeta_y=0$, $u'\xi_x+v'\xi_y=\xi u_x'+\eta v_x'$, $u'\eta_x+v'\eta_y=\xi u_y'+\eta v_y'$. On the other hand there are the conditions of integrability $u\zeta_x+v\zeta_y=0$, $u\xi_x+v\xi_y=\xi u_x+\eta v_x$, $u\eta_x+v\eta_y=\xi u_y+\eta v_y$. If ζ_x and ζ_y are not both zero, $u'/v'=u/v=f(x, y)$ and also $\eta=u_x=fv_x=-f\xi$. Hence $fv_x+v_y-vf_x$ is independent of x and it follows that u and v are independent of x.

A discussion of the two-dimensional case indicates that any irrotational motion can be superimposed upon a rotational motion with constant vorticity. The cases are found in which ζ is not constant. The case $\zeta_x=\zeta_y=0$ implies that the stream lines in both motions are straight with constant streaming. *H. Bateman* (Pasadena, Calif.).

Piskunov, N. S. On the problem of flow separation in a viscous fluid. C. R. (Doklady) Acad. Sci. URSS (N.S.) **37**, 43–45 (1942). [MF 8496]

The author investigates the conditions under which von Mises' boundary layer equation (that is, $z_{yy}=\nu^{-1}\theta^{-1}(x)z^{-1}z_x$ with boundary conditions $z(x, 0)=U^2(x)=\theta(x)$, $z(0, y)=\phi(y)$, $z(x, \infty)=0$) admits a back flow. In a previous paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **27**, 104–106 (1940); these Rev. **2**, 203] the author showed that in the case of nondecreasing $\theta(x)$ no back flow can occur. He then introduces two equations M_k, $k=1, 2$, which are similar to von Mises' equation but linear and states the following theorems. Assuming that M_1 has a solution possessing certain properties, then for certain functions $\theta(x)$ a back flow takes place. The second theorem is similar. (The occurrence of a back flow implies the existence of a separation point.) Additional results concerning flows with two separation points and a branch point are obtained. No proofs of the statements are given. *S. Bergman.*

Müller, Wilhelm. Über die Drehung zweier Zylinder in einer zähen Flüssigkeit und die Theorie der Kräfte am Rotationsviskosimeter mit exzentrischen Zylindern. Ann. Physik (5) **41**, 335–354 (1942). [MF 9233]

Müller, Wilhelm. Beitrag zur Theorie der langsamen Drehung zweier exzentrischer Kreiszylinder in der zähen Flüssigkeit. Z. Angew. Math. Mech. **22**, 177–189 (1942). [MF 8919]

The author continues his evaluation and discussion of the known solution of the two-dimensional problem of the slow motion of a viscous fluid between two rotating excentric cylinders, which he began in Ing.-Arch. **13**, 37–58 (1942); cf. these Rev. **5**, 21. *E. Reissner* (Cambridge, Mass.).

Kuo, Yung-Huai. On the force and moment acting on a body in shear flow. Quart. Appl. Math. **1**, 273–275 (1943). [MF 9368]

This note is supplementary to a recent paper by Tsien [same Quart. **1**, 130–148 (1943); these Rev. **5**, 21]. Beginning with the Eulerian dynamical equations, general expressions are derived for the force and moment on a body in a steady two-dimensional flow of constant vorticity distribution. These expressions are contour integrals involving the vorticity k and the derivative of the complex potential of the irrotational part of the flow. For $k=0$ they reduce to the well-known Blasius formulas for irrotational flow.
W. R. Sears (Inglewood, Calif.).

Krahn, E. Berechnung der zweiten Näherung der kompressiblen Strömung um ein Profil nach Janzen-Rayleigh. Luftfahrtforschung **20**, 147–151 (1943). [MF 9125]

The velocity potential of the two-dimensional irrotational flow of a compressible fluid around a profile is expanded in a power series progressing according to even powers of the Mach number. As is well known, the first term φ_0 of this development represents the velocity potential of the flow of an incompressible fluid around the same profile, and the second term φ_1 satisfies the equation

$$\Delta\varphi_1=(M^2/U^2)[u^2u_x+uv(u_y+v_x)+v^2v_y],$$

where u and v are the velocity components of the incompressible flow, M is the Mach number, U the velocity of the undisturbed stream and the subscripts x and y indicate differentiation with respect to these coordinates. Although presented in a different way, the method of obtaining a particular integral of this equation discussed in the present paper is essentially that of I. Imai and T. Aihara [Rep. Aeron. Res. Inst. Tokyo Imp. Univ. no. **199** (1940)]. Frequently only the velocities along the profile contour are wanted. The author shows that in this case it is sufficient to know the velocities of the incompressible flow along the contour, as well as the values which the mapping function and its normal derivative assume on the contour.
W. Prager (Providence, R. I.).

Tsien, Hsue-shen. On the design of the contraction cone for a wind tunnel. J. Aeronaut. Sci. **10**, 68–70 (1943). [MF 9220]

The problem is to design the contraction cone in such a way as to avoid adverse pressure gradients along the wall, which might lead to boundary layer separation or to shock waves. An arbitrary pressure distribution is assumed along the axis of the cone. The flow is then determined for the irrotational motion of a perfect incompressible fluid. This is accomplished by a separation of variables in cylindrical coordinates. The optimum shape of cone is given by the extreme stream line along which the pressure decreases monotonically. In practice, in order to provide a factor of safety, a stream line is chosen which allows an excess pressure gradient. *P. W. Ketchum* (Urbana, Ill.).

Szczeniowski, Boleslaw. Contraction cone for a wind tunnel. J. Aeronaut. Sci. **10**, 311–312 (1943). [MF 9219]

This paper is similar to that of Tsien reviewed above, except for the details of the calculations. Whereas Tsien starts with the error integral for the given velocity distribution along the axis of the cone, the present author uses the hyperbolic tangent. This leads to an expansion in Bessel functions rather than in Hermite polynomials; and an explicit expansion is found for the stream function, which Tsien obtains by numerical integration.
P. W. Ketchum (Urbana, Ill.).

Nikuradse, I. und Mohr, E. Zur Theorie des tragenden Flügels. Luftfahrtforschung **20**, 48–56 (1943). [MF 9044]

Considering the case of irrotational flow about an internal boundary in an infinite fluid, the authors determine the condition for minimum kinetic energy and given Kelvin impulse. This condition is found to be

$$\partial\varphi/\partial n=(\mathbf{w}+\mathbf{d}\times\mathbf{r})\cdot\mathbf{n},$$

where φ is the velocity potential, $\mathbf{n}$ is normal to the boundary, $\mathbf{w}$ and $\mathbf{d}$ are constants and $\mathbf{r}$ is the radius vector; that is, the flow is such as might be initiated by movement of the internal boundary as a rigid body. Such "rigid" flows are then classified according to the type of boundary involved, and the form of the impulse vectors is exhibited for each class. Proceeding to the case of a wing at rest in a uniform steady flow, the authors deduce the known ex-

pressions for the lift and drag in terms of the impulse and energy, respectively, in the two-dimensional flow far downstream. (The usual approximation is made here that the trailing vortices are straight and are parallel to the undisturbed velocity vector.) It follows that the minimum induced drag for a given lift and given dimensions corresponds to "rigid" flow in the wake. [Although this result was obtained by Munk from the Prandtl wing theory in 1919 [Inaug.-Dissertation, Göttingen; also Nat. Adv. Comm. Aeronaut. Rep. no. 121 (1921)] the present authors make no reference to the earlier paper or to any other published works. The authors' concluding recommendations to airplane designers indicate a lack of appreciation for the real problems of airfoil theory.] *W. R. Sears.*

Price, H. L. The lateral stability of aeroplanes. A new geometrical system of analysis. I. Aircraft Engrg. **15**, 193–198 (1943). [MF 8822]

The linearized differential equations for the motion of an airplane following a small disturbance from a steady glide are set up. The conditions for stability are noted and it is shown that the stability boundaries can be plotted on a diagram, which can also be used to indicate the amount of damping present. The diagrams are similar to those employed by Zimmerman [Nat. Adv. Comm. Aeronaut., Tech. Report no. **589** (1937)]. Further developments are to follow in subsequent papers. [Cf. the continuations reviewed below.] W. *R. Sears* (Inglewood, Calif.).

Price, H. L. The lateral stability of aeroplanes. A new geometrical system of analysis. II. Aircraft Engrg. **15**, 228–233 (1943). [MF 9128]

[Cf. the preceding and the following reviews.] In this paper the author develops formulae for the various derivatives that appear in the linearized equations of motion. He then evaluates these numerically for two classes of airplanes, "fighter class" and "bomber class," by inserting typical numerical values for some of the coefficients. Stability diagrams as described in part I can then be constructed for given angle of attack and radii of gyration for either of the classes mentioned. These diagrams have the static directional stability (μn_s) and the static lateral stability (μl_s) as coordinates and the "tail volume" as a parameter.
W. R. Sears (Inglewood, Calif.).

Price, H. L. The lateral stability of aeroplanes. A new geometrical system of analysis. III. Variation in "typical value" parameters. Aircraft Engrg. **15**, 265–269 (1943). [MF 9095]

[Cf. the preceding two reviews.] Here the author shows how to correct the stability diagrams constructed in part II for deviations of the coefficients from the typical values assumed there. He also discusses a geometrical construction that can be used to determine the complex roots of the stability quartic. *W. R. Sears* (Inglewood, Calif.).

Holt, Maurice. Calculation of wing profile drag. Aircraft Engrg. **15**, 278–280 (1943). [MF 9294]

Siestrunck, Raymond. Sur un mode de calcul de la distribution optima de circulation des hélices bipales. C. R. Acad. Sci. Paris **215**, 171–173 (1942). [MF 9484]

Malavard, Lucien. Pour le calcul des effets du fuselage et des fuseaux-moteurs sur la répartition en envergure des efforts aérodynamiques. C. R. Acad. Sci. Paris **215**, 266–268 (1942). [MF 9495]

Theory of Elasticity

Krzywoblocki, Zbigniew. The problem of wing oscillation. The influence of material and size on flexural and torsional frequencies. Aircraft Engrg. **15**, 234–237 (1943). [MF 8894]

The author uses for a wing model a uniform rod fixed at one end and vibrating freely under flexure or torsion. Using standard methods he calculates the smallest vibration frequency in these two cases and shows that they are proportional to (E/ρ) and (G/ρ), where E is the modulus of elasticity, G the modulus of rigidity and ρ the density of the material. A table of values of E/ρ and G/ρ for common structural materials is appended; values of the former show relatively little variation, but G/ρ for wood is only about one seventh that for metals. By dimensional arguments it is estimated that the frequencies will vary as $1/n$ if the dimensions of the wing are all multiplied by n. Since lower frequencies are deemed more serious, the opinion is expressed that wing oscillations will become more important in the design of future airplanes, particularly as size of airplanes increases. *P. W. Ketchum* (Urbana, Ill.).

Weber, Constantin. Eingrenzung von Verschiebungen mit Hilfe der Minimalsätze. Z. Angew. Math. Mech. **22**, 126–130 (1942). [MF 8893]

Weber, Constantin. Eingrenzung von Verschiebungen und Zerrungen mit Hilfe der Minimalsätze. Z. Angew. Math. Mech. **22**, 130–136 (1942). [MF 9534]

The author starts with the two well-known extremum principles of the theory of elasticity: (a) the principle of minimum total potential energy and (b) the principle of least work. These principles yield bounds on the work done by the external forces during the actual displacement if one arbitrarily assumes (a′) a displacement satisfying the boundary conditions but not necessarily the equilibrium conditions and (b′) a set of stresses satisfying the equilibrium conditions but not necessarily the boundary conditions, respectively. Bounds are thereby obtained for integrals involving the actual displacement. The problem in the present papers is to obtain bounds on the displacements themselves, rather than on their integrals. To accomplish this result the author assumes that one has independent knowledge of the approximate displacement in the neighborhood of a singularity in the external forces, that is, a concentrated load P at the point a in question. The principles (a) and (b) are applied to the sum of the given loading and the load P. (Actually the concentrated load P is spread uniformly over an infinitesimally small circle.) On combining the inequalities thus obtained, one arrives at two independent upper bounds for $w_a P/|P|$, where w_a is the actual displacement at a. Optimum values of P are determined by a quadratic equation with one positive and one negative root; these roots yield upper and lower bounds, respectively, for w_a. The final result is two upper and two lower bounds for the actual displacement. These bounds are made up of integrals which can be evaluated by assuming, for both the given loading and the load P, approximations to the displacement and stresses satisfying (a′) and (b′), respectively. If the assumed approximations are exact for either the given loading or the load P, the method gives the correct values of the displacement.

In the first paper the method is applied to the problem of the displacement at the center of a square elastic membrane with uniform transverse load, in the second paper to the general case of plane stress. In the latter case a distinc-

tion is made according to whether the point a is on the boundary or in the interior. The approximations corresponding to the load P are found from the Airy stress function for a half plane and whole plane, respectively. Bounds on the actual stresses are also found by an extension of the method, in which a concentrated load P at two neighboring points is considered. *P. W. Ketchum* (Urbana, Ill.).

Ament, W. S. The lines of principal stress in the plane problem of plasticity. Quart. Appl. Math. 1, 278–279 (1943). [MF 9370]

The author considers a plane plastic material which follows the Tresca yield condition. It is shown that, if $F(x, y)$ is the Airy stress function, then $z=\alpha F(x, y)$ is a Weingarten surface for which the difference of the principal curvatures is a constant (provided terms in α^2, α^3, etc. are neglected). Further, it is shown that the lines of curvature on a Weingarten surface, for which the difference of the principal curvatures is constant, form an equiareal pattern. It has been shown by Boussinesq [C. R. Acad. Sci. Paris 74 (1872)] and Sadowsky [J. Appl. Mech. 8, A-74–A-76 (1941); Amer. Math. Monthly 50, 35–40 (1943); these Rev. 3, 96 and 4, 179] that the lines of principal stress in the plastic plane form such an equiareal pattern. Thus, the author concludes that the orthogonal projection of the lines of curvature of the Weingarten surface on the xOy plane are the lines of principal stress. [Reviewer's note. The last statement may be rigorously proved by noting that the orthogonal projection of the lines of curvature satisfy, to within terms of order α^2, α^3, etc., the equation [Weatherburn, Differential Geometry, Cambridge University Press, Cambridge, England, 1927, p. 76]

$$(1)\qquad \frac{\partial^2 F}{\partial x\partial y}\left(\frac{dy}{dx}\right)^2+\left(\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial y^2}\right)\left(\frac{dy}{dx}\right)-\frac{\partial^2 F}{\partial x\partial y}=0.$$

Further, the lines of maximum shear stress are the characteristics of the yield condition [eq. 2 of the paper]. Hence, they satisfy

$$(2)\qquad \left(\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial y^2}\right)\left(\frac{dy}{dx}\right)^2-4\frac{\partial^2 F}{\partial x\partial y}\left(\frac{dy}{dx}\right)-\left(\frac{\partial^2 F}{\partial x^2}-\frac{\partial^2 F}{\partial y^2}\right)=0.$$

A simple computation reveals that the angle between the two sets of orthogonal directions defined by (1) and (2) is 45°. Hence, the equation (1) determines the lines of principal stress.] *N. Coburn* (Austin, Tex.).

Coburn, N. The linear yield condition in the plane plasticity problem. Duke Math. J. 10, 455–462 (1943). [MF 8966]

The author establishes geometrical properties of the lines of maximum shear stress in the problem of plane stress, where the principal normal stresses σ_1 and σ_2 satisfy the condition $\sigma_1-\sigma_2=b(\sigma_1+\sigma_2)+2c$, b and c constants. The results are interpretations of the differential equations for the coefficients of the line element, with the lines of maximum shear as parametric curves. *E. Reissner.*

Steuermann, E. Some special cases of the contact problem. C. R. (Doklady) Acad. Sci. URSS (N.S.) 38, 197–200 (1943). [MF 9243]

H. Hertz's classical treatment of the elastic contact problem is based on the assumption that the distance $d(x, y)$ between the two bodies in contact admits a development of the form $d=ax^2+2bxy+cy^2+\cdots$. In a recent paper [J. Appl. Math. Mech. [Akad. Nauk SSSR. Zhurnal Prikl. Mat. Mech.] 5, 409–418 (1941); these Rev. 5, 25] the author considered the case $d=(ax^2+by^2)^2+\cdots$. In this note he announces results for the case in which the second derivatives of d possess singularities at the origin (original point of contact). The following forms of d are considered: (a) $d=Ax^2+\cdots$ for $x\geqq 0$, $d=Bx^2+\cdots$ for $x\leqq 0$, $A\neq B$, (b) $d=A|x|^{\frac{1}{2}}+\cdots$, (c) $d=A(x^2+y^2)^{\frac{1}{2}}+\cdots$. The problem reduces to integral equations which the author solves in closed form, obtaining the dimensions of the contact surface and the pressure distribution. *L. Bers.*

Sherman, D. I. On the solution of a plane problem of the theory of elasticity for an isotropic medium. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 509–514 (1942). (Russian. English summary) [MF 8365]

Continuing his investigations on the solution of plane problems of elasticity [Scherman, C. R. (Doklady) Acad. Sci. URSS (N.S.) 27, 329–333, 911–913 (1940); 28, 25–27 (1940); these Rev. 2, 270] in multiply connected domains, the author solves the first problem of elasticity in an anisotropic medium by introducing the auxiliary variables $z_k=x+\mu_k y$, $k=1, 2$, where μ_k are suitably chosen constants. [The unknown functions then become analytic functions of the complex variable z_k. The idea of introducing z_k originates with S. G. Lekhnitsky, J. Appl. Math. Mech. [Akad. Nauk SSSR Zhurnal Prikl. Mat. Mech.] 2, 181–210 (1938).] Let S denote the multiply connected domain, S_j the holes, L_j their boundary curves, $j=1, \cdots, n$, and L_{n+1} the exterior boundary curve of S. Let the curves $L_j^{(k)}$ be the images of L_j and $S^{(k)}$ the images of S, $k=1, 2$. The first problem of elasticity reduces to the determination of two functions $\phi_k(z_k)$ which are regular in $S^{(k)}$ and satisfy on $L_j^{(k)}$ the boundary conditions

$$\Re\left[\sum_{k=1}^{2}(1+i\mu_k)\phi_k(t_k)\right]=f(t)+C_j,$$

where $f(t)$ is a known function and C_j constants. The author assumes that

$$\phi_1(z_1)=(2\pi i)^{-1}\int\omega(t)[t_1-z_1]^{-1}dt_1+\sum b_j(z_1-\alpha_j^{(1)})^{-1},\quad t=t(t_1),$$

and $\phi_2(z_2)$ is a similar expression. The integration is over $\sum L_j^{(1)}$; $\omega(t)$ is a new unknown function and α_j, b_j are constants which can be determined. Using the standard method he obtains for $\omega(t)$ the integral equation of Fredholm's type. The author shows that the homogeneous equation has only the trivial solution $\omega=0$, and therefore the nonhomogeneous equation always has a solution. The reviewer remarks that recently V. Morkovin applied the idea of Lekhnitsky to the study of the thin plate problem in an anisotropic medium [Cf. the following review.] *S. Bergman.*

Morkovin, Vladimir. On the deflection of anisotropic thin plates. Quart. Appl. Math. 1, 116–129 (1943). [MF 8803]

This paper deals with boundary problems of the differential equation

$$(*)\qquad w_{xxxx}+a_1w_{xxxy}+a_2w_{xxyy}+a_3w_{xyyy}+a_4w_{yyyy}=p.$$

The general solution of (*) is

$$w=f_1(z_1)+\bar{f}_1(\bar{z}_1)+f_2(z_2)+\bar{f}_2(\bar{z}_2)+w_p,$$

where $z_k=x+\mu_k y$ and where the complex constants μ_k are roots of the characteristic equation of (*). The author

restricts his discussion to boundary problems for the interior of simply connected regions. He assumes that the images in the z_k-planes of the boundary curve in the x,y-plane are mapped by functions $z_k=\sum_{-\infty}^{\infty}D_{km}\zeta_k^m$ on the circumference of unit-circles in the ζ_k-planes. He then substitutes in (*) $f_k(z_k)=\sum_{-\infty}^{\infty}A_{mk}\zeta_k^m$. The boundary conditions for w are rewritten in terms of the ζ_k-variables and become systems of equations for the coefficients A_{mk}. Further equations for the A_{mk} are derived from regularity conditions in the interior. As an application of the procedure the author obtains the solution for an elliptical plate with clamped edge and linearly varying load. Reference is made to earlier work of a related nature by S. S. Lechnitzky [Appl. Math. Mech. **2**, 181–210 (1938)]. *E. Reissner.*

Swoboda, H. Zum dreidimensionalen Spannungszustand der kreisrunden Platte. Z. Angew. Math. Mech. **20**, 336–350 (1940). [MF 8669]

The paper deals with the problem of the stresses in an elastic plate with the load uniformly distributed over a circular area the diameter of which is of the order of magnitude of the plate thickness. A successive approximation procedure is employed. Reference should be made to earlier solutions of this problem in exact form by A. Nadai [Elastische Platten, Springer, Berlin, 1924] and S. Woinowsky-Krieger [Ing.-Arch. **4**, 305 (1933)].

E. Reissner (Cambridge, Mass.).

Pöschl, Th. Über das Ziehen zäher Werkstoffe in konvergenten Kegeldüsen. I. Ing.-Arch. **13**, 175–184 (1942). [MF 8590]

Let r, θ and ϕ be spherical polar coordinates and take stresses and displacements independent of ϕ. The author shows that the following expressions for the displacement components are solutions of the theory of elasticity of an isotropic medium: $u_\phi=0$, $u_\theta=0$, $u_r=r^{-4(1-\nu)}g(\theta)+Ar^{-2}+Br$. In the expression for u_r, ν is Poisson's ratio, A and B are arbitrary constants and g is a solution of the equation

$$g''+\cot\theta g'+4(1-\nu)(5-4\nu)g=0.$$

This result is applied to the study of the stresses in the frustrum of a cone of opening angle $\theta=\alpha$. It is assumed that a stress resultant acts over the smaller of the two end sections of the solid, which stress resultant is balanced by the shear and normal stresses acting on the walls of the frustrum. *E. Reissner* (Cambridge, Mass.).

Sonntag, Rudolf. Die Kreisringfeder. Zur Theorie des geschlossenen Kreisringes mit grosser Formänderung. Ing.-Arch. **13**, 380–397 (1943). [MF 9076]

In considering large deflections of thin circular rings which are loaded along a diameter, the author employs the exact expressions for curvatures which yields results in terms of elliptical integrals. In particular, critical moments and displacements under the point compressive loads are found. Three cases arise under oppositely directed compressive loads: (a) $0\leqq\gamma\leqq0.63$, (b) $0.63\leqq\gamma<2.78$, (c) $\gamma\geqq2.78$, where $\gamma=QR_0/EI$, with Q equal to the load, R_0 the radius of undeformed ring, E the compressive modulus and I the moment of inertia of ring section. In (a) the square of the curvature under the load is greater than Q/EI. In (b) this curvature is smaller than the critical value, and no inflection points arise. The results thus far duplicate those of Fairthorne [Philos. Mag. (7) **26**, 1006–1017 (1938)]. In case (c) when inflection points are impending, the author considers the ring compressed between two rigid flat plates and the applied load consists of two loads $Q/2$ applied at the impending inflection points. The deformed ring is made up of two straight sections conforming with the loading plates and two arc sections. The remarkable result is that there is a hyperbolic relationship between γ and deflection ratio of the loaded diameter. The paper concludes by giving an approximate solution of this problem by means of Castigliano's principle and a suitably chosen "equivalent" ring.

D. L. Holl (Ames, Iowa).

Lubkin, S. and Stoker, J. J. Stability of columns and strings under periodically varying forces. Quart. Appl. Math. **1**, 215–236 (1943). [MF 9363]

The problem of the stability of a column under the periodic compressive force $F(t)=P+H\cos wt$, applied at the ends, is considered. For certain values of H and w the column is stable even when P exceeds the Euler load (that is, the least compressive steady load for which the unbent column becomes instable). However, it is also true that for certain values of H and w the column becomes instable when P is very much less than the Euler load and in fact even when P is a tension. The stability of a string under a periodically varying tension is also considered.

The solution of the column problem under consideration is expressible in the form $\sum F_n(\theta)\sin n\xi$, where the summation is for $n\geqq1$ and where $F_n(\theta)$ are solutions of the Mathieu equation

$$d^2F/d\theta^2+(\alpha+\beta\cos\theta)F=0$$

for values of α and β depending on n. The variables θ and ξ are essentially time and distance, respectively. In an appendix the convergence of the series $\sum F_n(\theta)\sin n\xi$ is studied. The authors present graphically and tabularly the values of α and β for which the solutions of the Mathieu equation are bounded as $\theta\to\infty$ and the values for which the solutions are unbounded.

Strict stability, in the sense that a really stable physical situation is implied, is shown by the authors to necessitate consideration of damping forces. The modification made by the inclusion of a damping term in the Mathieu equation on the behavior of its solutions as α and β vary is therefore considered by the authors.

A very useful bibliography is given which should be most helpful in the further pursuit of problems involving stability under periodic forces. *N. Levinson* (Cambridge, Mass.).

Cox, H. L. and Smith, H. E. The buckling of grids of stringers and ribs. Proc. London Math. Soc. (2) **48**, 1–26 (1943). [MF 9109]

The grid consists of $m-1$ similar stringers of uniform section, each $2B$ long, spaced $2D/m$ apart and cross-connected by $n-1$ similar ribs of uniform section, each $2D$ long, spaced $2B/n$ apart. The ends of both stringers and ribs are connected to edge members having half the torsional rigidities of the corresponding main members, and these edge members are held in the original plane. The problem is to calculate the buckling load of such a grid under a uniform compression in the direction of the stringers. Due to the particular choice of the edge members, the problem is equivalent to that of a grid of infinite dimensions so that no particular boundary restrictions appear. The authors solved this problem by two methods. First each stringer or rib is considered as a member subjected to bending and torsional loads. Due to the small deflections at the beginning of buckling, the bending and the twisting of

the members can be considered separately. The condition at the junction points of stringers and ribs are then used to obtain the "buckling equation" giving the load parameter θ as function of other parameters of the problem:

$$\theta=(2B/n)(P/EI_s)^{\frac{1}{2}},$$

where P is the load on each stringer, E Young's modulus, I_s the moment of inertia of stringer. The second method is the well-known Rayleigh-Ritz energy method. Here the deflection specified is very general and is capable of representing the exact deflection. Hence the results obtained are exact and are shown to be the same as those of first method.

If θ_1 is the load parameter corresponding to inter-rib buckling, that is, with the ribs twisted but not bent, then it is desirable to make θ equal to θ_1 for maximum structural efficiency. For this condition, it is shown that the deflection form gives one half-wave across the width of the grid and $n-1$ half-waves down the length of the grid (one half-wave less than the number of sections into which the stringers are divided by ribs). J. L. Taylor [Aeronaut. Res. Comm., R. and M. no. 1679] obtained an approximate solution by distributing the stiffnesses of the stringers and ribs over the whole plane of the grid. This approximate solution is shown to correspond with the author's calculation if the deflection form is assumed to be a simple sine curve. The required rib stiffness predicted by the approximate solution for developing maximum structural efficiency is however too low. This is caused by the considerable deviation of the wave form from simple sine curve in such cases.

H. S. Tsien (Pasadena, Calif.).

Cox, H. L. and Smith, H. E. The buckling of a thin sheet transversely stiffened. Proc. London Math. Soc. (2) **48**, 27–34 (1943). [MF 9110]

A plane rectangular sheet is stiffened by a number of uniformly spaced ribs in the direction of the width. The edge ribs have half the torsional rigidity of the main ribs and are held in the original plane. The structure is loaded by a uniform compression in the direction of the length. The edges of the sheet and the ends of all the ribs are held in the original plane but not otherwise constrained. The method used and the results obtained are similar to those of another paper [cf. the preceding review]. Here again, for maximum structural efficiency, the ribs should be so proportioned as to give equal buckling loads for inter-rib buckling and buckling with bending of the ribs.

H. S. Tsien (Pasadena, Calif.).

Nemenyi, P. and Truesdell, C. A stress function for the membrane theory of shells of revolution. Proc. Nat. Acad. Sci. U. S. A. **29**, 159–162 (1943). [MF 8335]

The authors take the three stress resultants of membrane theory in the form $N_\phi=N_{\phi n}(\phi)\cos n\theta$, $N_\theta=N_{\theta n}(\phi)\cos n\theta$, $N_{\phi\theta}=N_{\phi\theta n}(\phi)\sin n\theta$, where θ represents the meridional angle of a surface of revolution. Corresponding expressions are taken for the body force terms in the equilibrium equations. They then show that $N_{\phi n}$, $N_{\theta n}$, $N_{\phi\theta n}$ may be expressed in terms of a stress function U_n which satisfies an equation of the form

$$U_n''+(n^2-1)(f''/f)U_n=F_n,$$

where F_n is expressed in terms of the body force components and $r=f(z)$ determines the shape of the shell surface. There are listed a number of shell types which become amenable to analysis due to the introduction of the stress function and its relatively simple differential equation. It would be of some interest to establish a connection between the present results and some results obtained by A. J. Pucher [Proc. 5th International Congress Appl. Mech. 1938, pp. 134–139] on the subject of stress functions in the membrane theory of thin shells. *E. Reissner* (Cambridge, Mass.).

Sen, Bibhutibhusan. Stresses due to forces and couples acting in the interior of an infinite elastic slab placed on a rigid foundation. Bull. Calcutta Math. Soc. **35**, 13–20 (1943). [MF 9372]

The paper is concerned with the determination of the stresses and displacements produced by a vertical force or a couple with vertical vector acting at a point inside an infinite elastic slab of uniform thickness which rests on and adheres to a prefectly rigid foundation. Integral representations of stresses and displacements are given, but the evaluation is not carried to a stage of immediate usefulness in applications. *W. Prager* (Providence, R. I.).

Osgood, W. R. The center of shear again. J. Appl. Mech. **10**, A-62–A-64 (1943). [MF 8423]

The center of shear is that point of the cross section at the free end of a cantilever beam at which a transverse load must be applied in order that no rotation of the free end may occur in its own plane. The author explains that differences between various formulas for the determination of this point may be attributed to the assumptions of different boundary conditions. He proposes that the center of shear be defined as that point of the cross section through which the load must act in order that no twist shall occur along a spanwise line through the point of load application.

E. Reissner (Cambridge, Mass.).

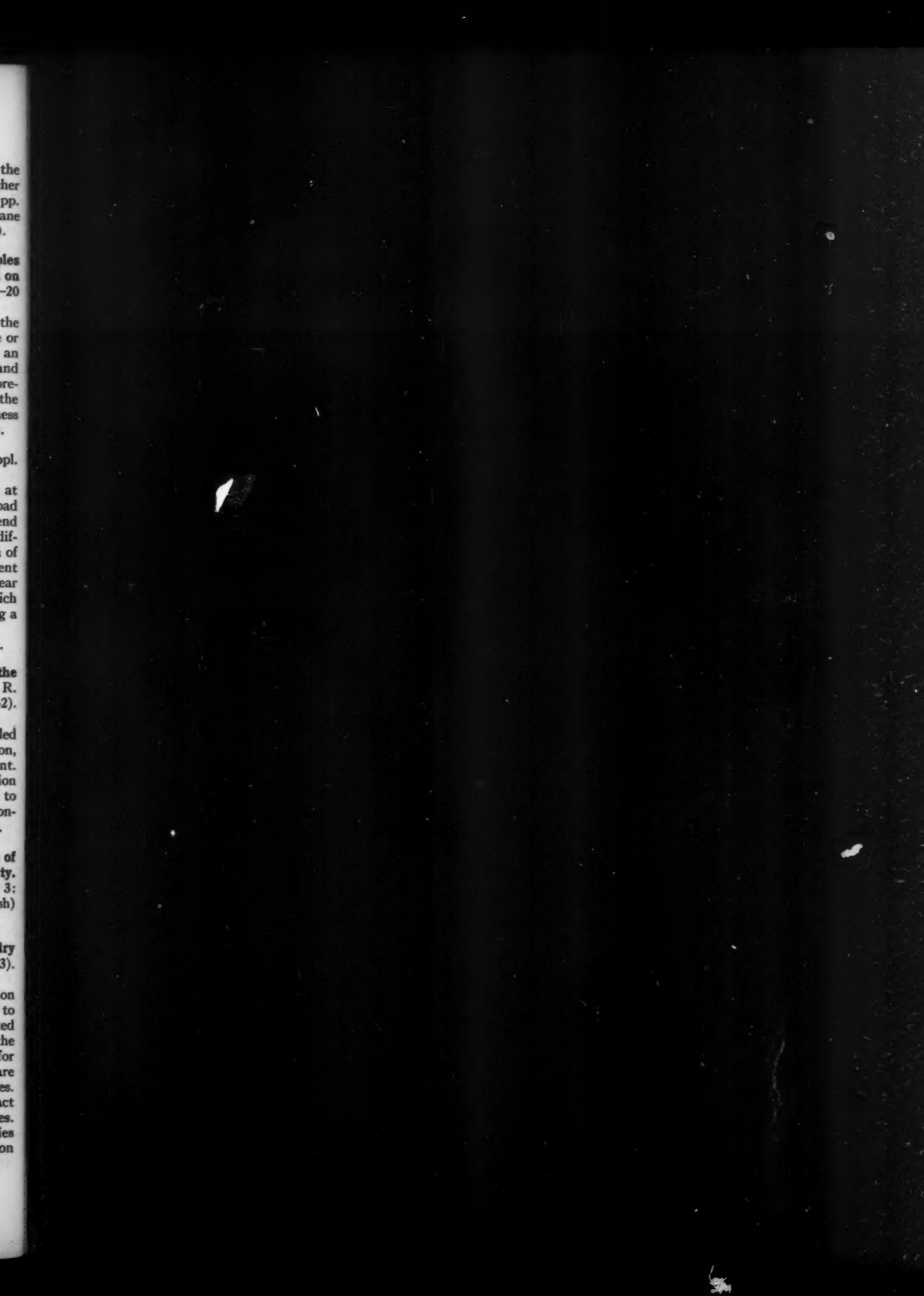
Sokolovsky, W. Elastico-plastic state of the tube in the presence of strain-hardening in the material. C. R. (Dokady) Acad. Sci. URSS (N.S.) **37**, 160–165 (1942). [MF 8657]

The paper is concerned with the stresses in a thick-walled tube under combined internal pressure and axial tension, part of the material being stressed beyond the yield point. H. Hencky's stress strain relations are used in connection with R. Schmidt's law of strain hardening. In contrast to the general practice in plasticity the material is not considered as incompressible. *W. Prager.*

Buich, Raul. Optical anisotropy for elastic deformation of transparent media and its application in photo-elasticity. Univ. Nac. La Plata. Publ. Fac. Ci. Fisicomat. Serie 3: Publ. Especiales no. **147**, 256 pp. (1941). (Spanish) [MF 9224]

Adirovich, E. and Blokhinzev, D. On the forces of dry friction. Acad. Sci. USSR. J. Phys. **7**, 29–36 (1943). [MF 8885]

The authors present a theory of dry or rubbing friction between bodies. The two bodies in contact are assumed to have similar protrusions which are regularly distributed along boundaries and form a doubly periodic surface, the size of the periods being the same for both bodies and for both the x and y directions. The equations of elasticity are solved subject to contact between the corrugated surfaces. Elastic waves traveling away from the area of contact provide the energy loss due to the so-called frictional forces. As the velocity is increased, the energy traveling out varies so as to provide a falling characteristic for the friction forces. *H. Poritsky* (Schenectady, N. Y.).



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